Alea **5**, 47–64 (2009)

Hölderian invariance principle for linear processes

Mindaugas Juodis, Alfredas Račkauskas and Charles Suquet

Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-2006 Vilnius, Lithuania and Institute of Mathematics and Informatics, Akademijos str. 4, LT-08663, Vilnius, Lithuania
E-mail address: `Mindaugas.Juodis@mif.vu.lt`

Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-2006 Vilnius, Lithuania and Institute of Mathematics and Informatics, Akademijos str. 4, LT-08663, Vilnius, Lithuania
E-mail address: `alfredas.rackauskas@maf.vu.lt`

Laboratoire P. Painlevé, UMR 8524 CNRS Université Lille I, Bât M2, Cité Scientifique, F-59655 Villeneuve d'Ascq Cedex, France
E-mail address: `Charles.Suquet@math.univ-lille1.fr`

Abstract. Let $(\xi_n)_{n\geq 1}$ be the polygonal partial sums processes built on the linear processes $X_n = \sum_{i\geq 0} a_i\epsilon_{n-i}$, where $(\epsilon_i)_{i\in\mathbb{Z}}$ are i.i.d., centered and square integrable random variables with $\sum_{i\geq 0} a_i^2 < \infty$. We investigate functional central limit theorem for ξ_n in the Hölder spaces $\mathrm{H}_\alpha^o[0,1]$ of functions $x : [0,1] \to \mathbb{R}$ such that $|x(t+h) - x(t)| = o(h^\alpha)$ uniformly in t. When $\sum_{i\geq 0} |a_i| < \infty$ (short memory case), we show that $n^{-1/2}\xi_n$ converges weakly in $\mathrm{H}_\alpha^o[0,1]$ to some Brownian motion under the optimal assumption that $P(|\epsilon_0| \geq t) = o(t^{-p})$, where $1/p = 1/2 - \alpha$. This extends the Lamperti invariance principle for i.i.d. X_n's. When $a_i = \ell(i)i^{-\beta}$, $1/2 < \beta < 1$, with ℓ positive, increasing and slowly varying, $(X_n)_{n\geq 1}$ has long memory. The limiting process for ξ_n is then the fractional Brownian motion W^H with Hurst index $H = 3/2 - \beta$ and the normalizing constants are $b_n = c_\beta n^H \ell(n)$. For $0 < \alpha < H - 1/2$, the weak convergence of $b_n^{-1}\xi_n$ to W^H in $\mathrm{H}_\alpha^o[0,1]$ is obtained under the mild assumption that $\mathbf{E}\,\epsilon_0^2 < \infty$. For $H - 1/2 < \alpha < H$, the same convergence is obtained under $P(|\epsilon_0| \geq t) = o(t^{-p})$, where $1/p = H - \alpha$.

1. Introduction

In the classical time series analysis, the innovations in the linear process $(X_n)_{n\in\mathbb{N}}$ are often assumed to be i.i.d. In this case asymptotic behaviors of the sample means

Received by the editors September 9 2007, accepted December 4 2007.
2000 *Mathematics Subject Classification.* 60F17, 62M10.
Key words and phrases. fractional Brownian motion, Hölder space, invariance principle, short memory, long memory, linear process, Wiener measure.
Research supported by a French-Lithuanian cooperation agreement "PAI Egide Gilibert".

and partial sum processes have been extensively studied in the literature. It would be hard to compile a complete list. Here we only mention some representatives: Davydov (1970), Gorodetskii (1977), Hall and Heyde (1980), Philips and Solo (1992) and Hosking (1996). See references therein for further background. There are basically two types of results. If the linear filter is absolutely summable, then the covariances of (X_n) are summable and one says that (X_n) is short-range dependent (SRD). Under SRD, the normalizing constant for the sum $S_n := X_1 + \cdots + X_n$ is of the same order as that in the classical CLT for i.i.d. observations. When the linear filter is not summable, then (X_n) is long-range dependent (LRD) and the normalizing constant for S_n is typically larger than square root of n. Fractional ARIMA model (Hosking, 1981) is an important class which may exhibit LRD. For a survey of LRD, we refer to Doukhan et al. (2003).

Invariance principles (or functional central limit theorems) play an important role in econometrics and statistics Stock (1994). For example, to obtain asymptotic distributions of unit root test statistics, researchers have applied invariance principles of various forms; see Sowell (1990) and Wu (2006) among others.

There is a large amount of papers which provide invariance principles for various linear processes in the framework of the classical function spaces, i.e. the space $\mathrm{C}[0,1]$ of continuous functions or the Skorokhod space $\mathrm{D}[0,1]$ of càdlàg functions. Our current contribution aims to investigate invariance principle for linear processes *in spaces having a stronger topology* than $\mathrm{C}[0,1]$.

The weak convergence of a sequence of stochastic processes in some functions space F provides results about the asymptotic distribution of functionals of the paths which are *continuous* with respect to the topology of F. Since the Hölder spaces are topologically embedded in $\mathrm{C}[0,1]$ and in $\mathrm{D}[0,1]$, they support more continuous functionals. From this point of view, the alternative framework of Hölder spaces gives functional limit theorems of a wider scope. This choice may be relevant as soon as the paths of stochastic processes and the limit process ξ (like e.g. the Brownian motion and the Fractional Brownian motion) share some Hölder regularity, roughly speaking $\xi(t+h) - \xi(t) = O(h^\alpha)$ for some $0 < \alpha < 1$. The first result in this direction seems to be Lamperti's Hölderian invariance principle Lamperti (1962) for the polygonal partial sums process $n^{-1/2}\xi_n$, where ξ_n is the polygonal line indexed by $[0,1]$ with vertices $(k/n, S_k)$, $k = 0, 1, \dots, n$ and the underlying random variables X_i are i.i.d. with $\mathbf{E}\, X_1 = 0$ and $\mathbf{E}\,|X_1|^q < \infty$ for some $q > 2$. This invariance principle was extended under some weak-dependance assumptions on the X_i's by Hamadouche (2000). Both results cost a stronger moment assumption than the classical square integrability of the X_i's, which is necessary and sufficient in the $\mathrm{C}[0,1]$ framework. Račkauskas and Suquet (2004b), found the right price to be paid to obtain an Hölderian invariance principle. They proved that for $0 < \alpha < 1/2$, $n^{-1/2}\xi_n$ converges in distribution to a Brownian motion in the Hölder space $\mathrm{H}_\alpha^o[0,1]$ (precise definition is given below) *if and only if*

$$\lim_{t\to\infty} t^{p(\alpha)} P(|X_1| > t) = 0, \quad \text{where } p(\alpha) = \frac{1}{\frac{1}{2} - \alpha}. \tag{1.1}$$

Contrastly Račkauskas and Suquet (2001) show how one can relax (1.1) in $\mathbf{E}\, X_1^2 < \infty$ by using selfnormalization and adaptive construction of the partial sums process. These theoretical results found statistical applications in the problem of detection of a changed segment in data Račkauskas and Suquet (2004a, 2006).

For recent result and a survey in the domain of the invariance principles for the linear processes we refer to Merlevède et al. (2006) and Peligrad and Utev (2005, 2006a,b). These papers fully analyze the asymptotic properties of the partial sums of the linear process, and extend the results for various noise processes, in the framework of the spaces $\mathrm{D}[0,1]$ or $\mathrm{C}[0,1]$. The same holds for other approaches involving invariance principles for the linear processes (see Wu and Min, 2005; Wu, 2006 with comprehensive list of bibliography).

In this paper we consider the polygonal partial sums processes $(\xi_n)_{n\geq 1}$ built on the linear processes $X_n = \sum_{i\geq 0} a_i\epsilon_{n-i}$, where $(\epsilon_i)_{i\in\mathbb{Z}}$ are i.i.d., centered and square integrable random variables with $\sum_{i\geq 0} a_i^2 < \infty$. We investigate functional central limit theorem for ξ_n in the Hölder spaces $\mathrm{H}_\alpha^o[0,1]$. When $\sum_{i\geq 0}|a_i| < \infty$ (short memory case), we show that $n^{-1/2}\xi_n$ converges weakly in $\mathrm{H}_\alpha^o[0,1]$ to some Brownian motion under the optimal assumption that $P(|\epsilon_0| \geq t) = o(t^{-p})$, where $1/p = 1/2-\alpha$. This extends the Lamperti invariance principle for i.i.d. X_n's. When $a_i = \ell(i)i^{-\beta}$, $1/2 < \beta < 1$, with ℓ positive, increasing and slowly varying, $(X_n)_{n\geq 1}$ has long memory. The limiting process for ξ_n is then the fractional Brownian motion W^H with Hurst index $H = 3/2 - \beta$ and the normalizing constants are $b_n = c_\beta n^H \ell(n)$. For $0 < \alpha < H - 1/2$, the weak convergence of $b_n^{-1}\xi_n$ to W^H in $\mathrm{H}_\alpha^o[0,1]$ is obtained under the mild assumption that $\mathbf{E}\,\epsilon_0^2 < \infty$, supplementing Wu and Min (2005) invariance principle in $C[0,1]$. For $H - 1/2 < \alpha < H$, the same convergence is obtained under $P(|\epsilon_0| \geq t) = o(t^{-p})$, where $1/p = H - \alpha$. The case $\alpha = H - 1/2$ is also discussed.

The paper is organized as follows. Section 2 gives the notations and results. Section 3 presents the proofs, starting with a general theorem on Hölderian invariance principles for dependent variables which enables us to simplify the proofs of our main results. It may be also of independent interest. Technical lemmas are gathered in Section 4.

2. Results

2.1. *Notations.* For $0 < \alpha < 1$, we denote by $\mathrm{H}_\alpha^o[0,1]$ the set of real valued continuous functions $x : [0,1] \to \mathbb{R}$ such that

$$\lim_{\delta\to 0} w_\alpha(x,\delta) = 0,$$

where

$$w_\alpha(x,\delta) = \sup_{0<|t-s|<\delta} \frac{|x(t)-x(s)|}{|s-t|^\alpha}.$$

The set $\mathrm{H}_\alpha^o[0,1]$ is a separable Banach space when endowed with the norm $||x||_\alpha = |x(0)| + w_\alpha(x,1)$. Let ξ_n $(n \geq 1)$ and ξ be random elements in $\mathrm{H}_\alpha^o[0,1]$. The weak convergence *in* $\mathrm{H}_\alpha^o[0,1]$ of ξ_n to ξ, denoted by

$$\xi_n \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^o} \xi$$

means that for every fonctional $f : \mathrm{H}_\alpha^o[0,1] \to \mathbb{R}$, continuous with respect to the strong topology of $\mathrm{H}_\alpha^o[0,1]$, the sequence of random variables $f(\xi_n)$ converges to $f(\xi)$ in distribution.

For the sequence $(X_n)_{n\geq 1}$ of random variables, put

$$S_0 := 0, \quad S_n := \sum_{i=1}^{n} X_i \tag{2.1}$$

and define the partial sums process ξ_n by

$$\xi_n(t) := S_{[nt]} + (nt - [nt])X_{[nt]+1}, \quad t \in [0,1], \tag{2.2}$$

where $[nt]$ denotes the integer part of nt. As polygonal lines, the paths of ξ_n belong to $\mathrm{H}_\alpha^o[0,1]$ for every $\alpha < 1$.

Recall that the standard fractional Brownian motion W_H, with the Hurst index H is a zero mean Gaussian process with covariance

$$\mathbf{E}\, W_H(t)W_H(s) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \quad 0 \leq s,t \leq 1.$$

The special case $H = 1/2$ gives the Brownian motion denoted W. The limiting processes ξ involved in this paper are either W, either W_H with positively correlated increments, that is $H > 1/2$. Almost all paths of W_H are Hölder continuous of any order α strictly less than H.

The linear processes $(X_k)_{k\geq 0}$ considered throughout the paper are of the form

$$X_k = \sum_{i=0}^{\infty} a_i \epsilon_{k-i}, \quad k = 0,1,\ldots, \tag{2.3}$$

where $(a_i, i \in \mathbb{Z})$ is a given sequence of real numbers with $a_i = 0$ for $i < 0$ and $(\epsilon_i, i \in \mathbb{Z})$ is a sequence of independent identically distributed random variables with $\mathbf{E}\,\epsilon_0 = 0$ and $\mathbf{E}\,|\epsilon_0|^2 < \infty$. Under these assumptions, the series in (2.3) converges in L^2 and almost surely and the sequence of random variables $(X_k)_{k\geq 0}$ is stationary.

2.2. *Linear processes with short memory.*

Theorem 2.1. *Let $(X_k)_{k\geq 0}$ be the linear process defined by* (2.3) *and assume that $(a_i)_{i\geq 0}$ satisfies:*

$$(A) \quad \sum_{i=0}^{\infty} |a_i| < \infty \quad \text{and} \quad A := \Big|\sum_{i=0}^{\infty} a_i\Big| > 0.$$

Let S_n and ξ_n be the partial sums and partial sums process built on $(X_k)_{k\geq 0}$, defined by (2.1) *and* (2.2). *Put $b_n^2 = A^2 n \mathbf{E}\,\epsilon_0^2$, $b_n > 0$. Then for every $0 < \alpha < 1/2$,*

$$b_n^{-1}\xi_n \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^o} W$$

if

$$\lim_{t\to\infty} t^p P(|\epsilon_0| > t) = 0, \quad \text{where } p = \frac{1}{\frac{1}{2} - \alpha}. \tag{2.4}$$

Condition (2.4) is optimal because the class of linear processes considered includes the special case where $X_k = \epsilon_k$ and it is known that in this case (2.4) is necessary for the weak-$\mathrm{H}_\alpha^o[0,1]$ convergence of $n^{-1/2}\xi_n$ to W, see Račkauskas and Suquet (2004b).

2.3. *Linear processes with long memory.* Now we consider a class of linear processes whose associated partial sums process converges to a fractional Brownian motion W_H with $H > 1/2$.

Theorem 2.2. *For $1/2 < \beta < 1$, let $(X_k)_{k\geq 0}$ be the linear process*

$$X_k = \sum_{j=0}^{\infty} \psi_j \epsilon_{k-j}, \quad \text{with } \psi_0 = 1,\ \psi_j = \frac{\ell(j)}{j^{\beta}},\ j \geq 1, \tag{2.5}$$

where ℓ is a positive non decreasing normalized slowly varying function and $(\epsilon_j, j \in \mathbb{Z})$ is a sequence of i.i.d. random variables with $\mathbf{E}\,\epsilon_0 = 0$ and $\mathbf{E}\,|\epsilon_0|^2$ is finite. Put

$$H := \frac{3}{2} - \beta. \tag{2.6}$$

Let S_n and ξ_n be the partial sums and partial sums process built on $(X_k)_{k\geq 0}$, defined by (2.1) and (2.2). Put

$$b_n = n^H \ell(n) c_\beta \big(\mathbf{E}\,\epsilon_0^2\big)^{1/2}, \tag{2.7}$$

with

$$c_\beta := (1-\beta)^{-2} \int_0^{\infty} \big(x^{1-\beta} - (x-1)_+^{1-\beta}\big)^2 \,\mathrm{d}x, \quad \text{where } x_+ := \max(0; x).$$

Then for $0 < \alpha < H$, the weak-Hölder convergence

$$b_n^{-1} \xi_n \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^o} W_H \tag{2.8}$$

is obtained in the following cases.

(1) *For $0 < \alpha < H - 1/2$, (2.8) holds true if $\mathbf{E}\,\epsilon_0^2 < \infty$.*
(2) *For $\alpha = H - 1/2$, (2.8) holds true if*

$$\lim_{t\to\infty} (t \ln t)^2 P(|\epsilon_0| > t) = 0 \tag{2.9}$$

(3) *For $H - 1/2 < \alpha < H$, (2.8) holds true if*

$$\lim_{t\to\infty} t^p P(|\epsilon_0| > t) = 0, \quad \text{where} \quad p = \frac{1}{H-\alpha}. \tag{2.10}$$

The slowly varying function ℓ is said normalized if for every δ positive, $t^{\delta}\ell(t)$ is ultimately increasing and $t^{-\delta}\ell(t)$ is ultimately decreasing.

The variance σ_n^2 of S_n is asymptotically equivalent to b_n^2, see Wu and Min (2005, Th.2). Therefore the convergence (2.8) holds as well with b_n replaced by σ_n.

The necessity of condition (2.10) remains an open question. To our best knowledge necessary moment conditions for limit behavior of sums of long memory linear processes are not treated in literature.

Another interesting open problem was pointed out by the Referee, namely, the case $\beta = 1/2$ in Theorem 2.2. Does the convergence to Brownian motion still holds provided $\ell(n)$ does not have subsequence tending to zero? At the moment we have no answer to this question.

3. Proofs

3.1. *General reduction.* We describe here the common part of the proofs of Theorems 2.1 and 2.2 which provides a general methodology to establish the weak-$\mathrm{H}^o_\alpha[0,1]$ convergence of the partial sums process. This may be of independent interest to prove invariance principles under various kind of dependence of the underlying sequence $(X_n)_{n\geq 1}$. Classically $b_n^{-1}\xi_n$ converges weakly to ξ in $\mathrm{H}^o_\alpha[0,1]$ if and only if

a) the finite dimensional distributions of $b_n^{-1}\xi_n$ converge to those of ξ;
b) the sequence $(b_n^{-1}\xi_n)_{n\geq 1}$ is tight in $\mathrm{H}^o_\alpha[0,1]$.

Usually condition a) is known to be satisfied under mild assumptions, e.g. if weak convergence of $b_n^{-1}\xi_n$ is already established in $C[0,1]$. This is indeed the case in the context of Theorems 2.1 and 2.2. So we will focuse on the tightness problem. General conditions implying the tighness of a sequence of random elements in $\mathrm{H}^o_\alpha[0,1]$ may be found in Račkauskas and Suquet (2001) (Prop. 7 and Rem. 8). To translate this result in the setting of partial sums process ξ_n, write for simplicity

$$t_k = t_{j,k} = k2^{-j}, \qquad k = 0,1,\dots,2^j, \quad j = 1,2,\dots$$

Then the tighness of $(b_n^{-1}\xi_n)_{n\geq 1}$ in $\mathrm{H}^o_\alpha[0,1]$ takes place provided that

i) for every $t\in[0,1]$, $(b_n^{-1}\xi_n(t))_{n\geq 1}$ is tight on $\mathbb{R}$;
ii) $\lim\limits_{J\to\infty}\limsup\limits_{n\to\infty} P\Big\{\sup\limits_{j\geq J} 2^{j\alpha} b_n^{-1}\max\limits_{0\leq k<2^j}|\xi_n(t_{k+1})-\xi_n(t_k)|\geq\varepsilon\Big\}=0.$

Now we are able to go a step further by proving the following theorem. It is worth noticing that *nothing is assumed about the dependence structure* of $(X_n)_{n\geq 1}$ in its statement.

Theorem 3.1. *Let ξ_n be the partial sums process built on $(X_k)_{k\geq 0}$, defined by (2.2). Then $(b_n^{-1}\xi_n)_{n\geq 1}$ is tight in $\mathrm{H}^o_\alpha[0,1]$ if:*

(1) *for every $t\in[0,1]$, $(b_n^{-1}\xi_n(t))_{n\geq 1}$ is tight on $\mathbb{R}$;*
(2) *$n^\alpha b_n^{-1}\max\limits_{1\leq i\leq n}|X_i|$ converges in probability to 0;*
(3) $\lim\limits_{J\to\infty}\limsup\limits_{n\to\infty} P\Big\{\max\limits_{J\leq j\leq\log n} 2^{j\alpha} b_n^{-1}\max\limits_{0\leq k<2^j}|S_{[nt_{k+1}]}-S_{[nt_k]}|\geq\varepsilon\Big\}=0$
for every positive ε.

Here and throughout the paper, $\log n$ stands for the logarithm with basis 2, so that $2^{\log n}=n$. The following corollary suits better our needs.

Corollary 3.2. *Assume that the X_i's have identical distribution. Then $(b_n^{-1}\xi_n)_{n\geq 1}$ is tight in $\mathrm{H}^o_\alpha[0,1]$ if Conditions 1 and 3 of Theorem 3.1 are satisfied and*

$$\forall\varepsilon>0,\quad nP(|X_1|\geq\varepsilon b_n n^{-\alpha})\xrightarrow[n\to\infty]{}0. \tag{3.1}$$

Clearly under identical distribution of the X_i's, (3.1) implies Condition 2 in Theorem 3.1. Moreover when (3.1) is enough for $(b_n^{-1}\xi_n)_{n\geq 1}$ to satisfy the invariance principle in C$[0,1]$, then we can drop Condition 1 and concentrate on the verification of (3.1) and Condition 3 to prove the invariance principle in $\mathrm{H}^o_\alpha[0,1]$.

Proof of Theorem 3.1. We have to check ii). Denote by $P_0=P_0(J,n)$ the probability appearing in Condition ii). Then P_0 is bounded by P_1+P_2 where

$$P_1 := P\Big\{\max_{J\leq j\leq\log n} 2^{j\alpha} b_n^{-1}\max_{0\leq k<2^j}|\xi_n(t_{k+1})-\xi_n(t_k)|\geq\varepsilon\Big\}$$

and

$$P_2 := P\Big\{\sup_{j>\log n} 2^{j\alpha} b_n^{-1} \max_{0\le k<2^j} |\xi_n(t_{k+1}) - \xi_n(t_k)| \ge \varepsilon\Big\}.$$

Estimation of P_2. As $j > \log n$, $t_{k+1} - t_k = 2^{-j} < 1/n$ and then with t_k in say $[i/n, (i+1)/n)$, either t_{k+1} is in $(i/n, (i+1)/n]$ or belongs to $\big((i+1)/n, (i+2)/n\big]$, where $1 \le i \le n-2$ depends on k and j.

In the first case, noting that the slope of ξ_n on $[i/n, (i+1)/n)$ is exactly nX_{i+1}, we have

$$|\xi_n(t_{k+1}) - \xi_n(t_k)| = n|X_{i+1}|2^{-j} \le 2^{-j} n \max_{1\le i\le n} |X_i|.$$

If t_k and t_{k+1} are in consecutive intervals, then

$$\begin{aligned}|\xi_n(t_{k+1}) - \xi_n(t_k)| &\le |\xi_n(t_k) - \xi_n((i+1)/n)| + |\xi_n((i+1)/n) - \xi_n(t_{k+1})|\\ &\le 2^{-j+1} n \max_{1\le i\le n} |X_i|.\end{aligned}$$

With both cases taken into account we obtain

$$\begin{aligned}P_2 &\le P\Big\{\sup_{j>\log n} 2^{j\alpha} b_n^{-1} n 2^{-j+1} \max_{1\le i\le n} |X_i| \ge \varepsilon\Big\}\\ &= P\Big\{n b_n^{-1} \max_{1\le i\le n} |X_i| \sup_{j>\log n} 2^{(\alpha-1)j} \ge \frac{\varepsilon}{2}\Big\}\\ &\le P\Big\{n^\alpha b_n^{-1} \max_{1\le i\le n} |X_i| \ge \frac{\varepsilon}{2}\Big\},\end{aligned}$$

so by Condition 2, $\lim_{n\to\infty} P_2 = 0$.

Estimation of P_1. Let $u_k = [nt_k]$. Then $u_k \le nt_k \le 1 + u_k$ and $1 + u_k \le u_{k+1} \le nt_{k+1} \le 1 + u_{k+1}$. Therefore

$$|\xi_n(t_{k+1}) - \xi_n(t_k)| \le |\xi_n(t_{k+1}) - S_{u_{k+1}}| + |S_{u_{k+1}} - S_{u_k}| + |S_{u_k} - \xi_n(t_k)|.$$

Since $|S_{u_k} - \xi_n(t_k)| \le |X_{1+u_k}|$ and $|\xi_n(t_{k+1}) - S_{u_{k+1}}| \le |X_{1+u_{k+1}}|$ we obtain $P_1 \le P_{1,1} + P_{1,2}$, where

$$\begin{aligned}P_{1,1} &:= P\Big\{\max_{J\le j\le \log n} 2^{j\alpha} b_n^{-1} \max_{1\le k\le 2^j} |S_{u_{k+1}} - S_{u_k}| \ge \frac{\varepsilon}{2}\Big\}\\ P_{1,2} &:= P\Big\{\max_{J\le j\le \log n} 2^{j\alpha} b_n^{-1} \max_{1\le i\le n} |X_i| \ge \frac{\varepsilon}{4}\Big\}.\end{aligned}$$

In $P_{1,2}$, the maximum over j is realized for $j = [\log n]$, so $\lim_{n\to\infty} P_{1,2} = 0$ by Condition 2.

Gathering all the estimates, we finally obtain

$$\lim_{J\to\infty} \limsup_{n\to\infty} P_0 = \lim_{J\to\infty} \limsup_{n\to\infty} P_{1,1} = 0,$$

by Condition 3. □

We now turn to the proofs of Theorems 2.1 and 2.2. To avoid disturbing the main flow of argumentation, we deferred technical lemmas to subsection 3.4.

3.2. *Short memory.*

Proof of Theorem 2.1. We need to check the convergence of finite dimensional distributions and tightness. Put $\sigma_n^2 := \mathbf{E}\, S_n^2$. By a classical computation

$$\frac{\sigma_n^2}{n} = \mathbf{E}\,\epsilon_0^2 \sum_{i,k=0}^{\infty} a_i a_k \Big(1 - \frac{|i-k|}{n}\Big)_+ .$$

Due to assumption (A), $\sum_{i,k=0}^{\infty} |a_i a_k|$ is finite, so by the bounded convergence theorem for the series

$$\frac{\sigma_n^2}{n} \xrightarrow[n\to\infty]{} \mathbf{E}\,\epsilon_0^2 \sum_{i,k=0}^{\infty} a_i a_k = A^2 \mathbf{E}\,\epsilon_0^2, \tag{3.2}$$

recalling that $A := \big|\sum_{i=0}^{\infty} a_i\big|$. In what follows we assume without loss of generality that $\mathbf{E}\,\epsilon_0^2 = 1$. As b_n and σ_n are asymptotically equivalent, the $\mathrm{C}[0,1]$ or $\mathrm{H}_\alpha^o[0,1]$ convergences of $b_n^{-1}\xi_n$ and $\sigma_n^{-1}\xi_n$ are equivalent. The convergence of the finite dimensional distributions of $b_n^{-1}\xi_n$ to those of the standard Brownian motion W follows of the weak convergence in $\mathrm{C}[0,1]$ of $\sigma_n^{-1}\xi_n$ to W. Such an invariance principle may be found for instance in Wu and Min (2005), Theorem 1. That theorem involves more general linear filters and condition (A) is just a special case (see also Remark 4 in Wu and Min, 2005). As a by-product of this invariance principle, Condition 1 in Theorem 3.1 is automatically satisfied.

To check the tightness, we use Corollary 3.2. First we note that our assumption (2.4) implies via Lemma 3.7 below that

$$\lim_{t\to\infty} t^p P(|X_0| \geq t) = 0.$$

As $b_n = An^{1/2}$ and $1/p = 1/2 - \alpha$, we deduce immediately (3.1) from the above limit. So it remains only to check Condition 3 of Theorem 3.1, that is $\lim_{J\to\infty} \limsup_{n\to\infty} P_1(J,n,\varepsilon) = 0$, with

$$P_1(J,n,\varepsilon) = P\Big\{\max_{J\leq j\leq \log n} 2^{j\alpha} b_n^{-1} \max_{0\leq k<2^j} |S_{u_{k+1}} - S_{u_k}| \geq \varepsilon\Big\}, \tag{3.3}$$

where $u_k = [nt_k] = [nk2^{-j}]$.

Let us fix an arbitrary $\delta > 0$, put $\Delta_n := \delta n^{1/p}$ and define

$$\widehat{\epsilon_l} := \epsilon_l \mathbf{1}\{|\epsilon_l| \leq \Delta_n\} - \mathbf{E}\,\epsilon_l \mathbf{1}\{|\epsilon_l| \leq \Delta_n\}, \tag{3.4}$$

$$\widetilde{\epsilon_l} := \epsilon_l \mathbf{1}\{|\epsilon_l| > \Delta_n\} - \mathbf{E}\,\epsilon_l \mathbf{1}\{|\epsilon_l| > \Delta_n\}. \tag{3.5}$$

Since $\mathbf{E}\,\epsilon_l = 0$, $\epsilon_l = \widehat{\epsilon_l} + \widetilde{\epsilon_l}$ and we have

$$\sum_{i=u_k}^{u_{k+1}} X_i = \sum_{l=-\infty}^{\infty} \Bigg(\sum_{i=u_k}^{u_{k+1}} a_{i-l}\Bigg)\epsilon_l = Z_{j,k}^{(1)} + Z_{j,k}^{(2)},$$

where

$$Z_{j,k}^{(1)} = \sum_{l=-\infty}^{\infty} \Bigg(\sum_{i=u_k}^{u_{k+1}} a_{i-l}\Bigg)\widehat{\epsilon_l} \quad \text{and} \quad Z_{j,k}^{(2)} = \sum_{l=-\infty}^{\infty} \Bigg(\sum_{i=u_k}^{u_{k+1}} a_{i-l}\Bigg)\widetilde{\epsilon_l}. \tag{3.6}$$

Hence, we have to prove both

$$\lim_{J\to\infty} \limsup_{n\to\infty} P_1^{(i)}(J,n,\varepsilon) = 0, \quad i = 1,2, \tag{3.7}$$

where for $i = 1, 2$,

$$P_1^{(i)}(J, n, \varepsilon) := P\left\{ \max_{J \le j \le \log n} 2^{\alpha j} \max_{0 \le k < 2^j} \left| Z_{j,k}^{(i)} \right| > b_n \frac{\varepsilon}{2} \right\}.$$

To estimate $P_1^{(2)}(J, n, \varepsilon)$, first apply Chebyshev inequality to obtain

$$P_1^{(2)}(J, n, \varepsilon) \le \sum_{J \le j \le \log n} 2^{2\alpha j} b_n^{-2} 4\varepsilon^{-2} \sum_{0 \le k < 2^j} \mathbf{E} \left| Z_{j,k}^{(2)} \right|^2. \tag{3.8}$$

Next, observe that by stationarity, $\sum_{l=-\infty}^{\infty} \left| \sum_{i=u_k}^{u_{k+1}} a_{i-l} \right|^2 \mathbf{E}\, \epsilon_0^2 = \sigma_{u_{k+1}-u_k}^2$, whence it follows via (3.2) that for some constant c,

$$\sum_{l=-\infty}^{\infty} \left| \sum_{i=u_k}^{u_{k+1}} a_{i-l} \right|^2 \le c(u_{k+1} - u_k). \tag{3.9}$$

This gives

$$\mathbf{E} \left| Z_{j,k}^{(2)} \right|^2 = \sum_{l=-\infty}^{\infty} \left(\sum_{i=u_k}^{u_{k+1}} a_{i-l} \right)^2 \mathbf{E}\, |\widetilde{\epsilon}_l|^2 \le c(u_{k+1} - u_k) \mathbf{E}\, |\widetilde{\epsilon}_0|^2 \le 2n2^{-j} c \mathbf{E}\, |\widetilde{\epsilon}_0|^2.$$

Now using inequality (3.23) in Lemma 3.4 and recalling that $\Delta_n = \delta n^{1/p}$, $b_n^2 = A^2 n$ and $1/p = 1/2 - \alpha$, we obtain

$$\begin{aligned}
P_1^{(2)}(J, n, \varepsilon) &\le \frac{8cp\delta^{2-p}}{(p-2)\varepsilon^2} \sum_{J \le j \le \log n} 2^{2\alpha j} b_n^{-2} 2^j n 2^{-j} n^{2/p-1} \sup_{t \ge \Delta_n} t^p P(|\epsilon_0| > t) \\
&= \frac{8cp\delta^{2-p}}{(p-2)A^2\varepsilon^2} n^{2/p-1} \sup_{t \ge \Delta_n} t^p P(|\epsilon_0| > t) \sum_{j=J}^{\log n} 2^{2\alpha j} \\
&\le \frac{8cp\delta^{2-p}}{(p-2)A^2\varepsilon^2} n^{2/p-1} \sup_{t \ge \Delta_n} t^p P(|\epsilon_0| > t) \frac{2^{2\alpha} n^{2\alpha}}{2^{2\alpha} - 1} \\
&\le \frac{16cp\delta^{2-p}}{(p-2)A^2\varepsilon^2(2^{2\alpha} - 1)} \sup_{t \ge \Delta_n} t^p P(|\epsilon_0| > t).
\end{aligned}$$

Thus (2.4) gives

$$\lim_{n \to \infty} P_1^{(2)}(J, n, \varepsilon) = 0. \tag{3.10}$$

To estimate $P_1^{(1)}(J, n, \varepsilon)$, let us fix some $q > p$ and apply the Markov inequality of order q to start with:

$$P_1^{(1)}(J, n, \varepsilon) \le \frac{2^q}{\varepsilon^q b_n^q} \sum_{J \le j \le \log n} \sum_{0 \le k < 2^j} 2^{q\alpha j} \mathbf{E} \left| Z_{j,k}^{(1)} \right|^q. \tag{3.11}$$

By Rosenthal's inequality, see (3.19) in Lemma 3.3 below,

$$\mathbf{E} \left| Z_{j,k}^{(1)} \right|^q \le R_q \left(\sum_{l=-\infty}^{\infty} \left| \sum_{i=u_k}^{u_{k+1}} a_{i-l} \right|^2 \mathbf{E}\, |\widehat{\epsilon}_l|^2 \right)^{q/2} + R_q \sum_{l=-\infty}^{\infty} \left| \sum_{i=u_k}^{u_{k+1}} a_{i-l} \right|^q \mathbf{E}\, |\widehat{\epsilon}_l|^q. \tag{3.12}$$

As the series $\sum_{i=0}^{\infty} |a_i|$ converges, we have

$$A_0 := \sup_{I \subset \mathbb{N}} \left| \sum_{i \in I} a_i \right| < \infty.$$

Thus from (3.9) we get $\sum_{l=-\infty}^{\infty}\bigl|\sum_{i=u_k}^{u_{k+1}} a_{i-l}\bigr|^q \le cA_0^{q-2}(u_{k+1}-u_k) \le 2cA_0^{q-2}n2^{-j}$. From now on, we denote by C a constant which may depend of ε, q, α, c, A, A_0 and of the distribution of ϵ_0. Its explicit value is allowed to vary from one line to another. Going back to Rosenthal inequality with the above estimate and the inequalities (3.22) and (3.25), we get for n large enough:

$$\mathbf{E}\,\bigl|Z_{j,k}^{(1)}\bigr|^q \le C\bigl(n^{q/2}2^{-jq/2} + \delta^{q-p}n^{q/p}2^{-j}\bigr).$$

Thus we can bound $P_1^{(1)}(J,n,\varepsilon)$ by

$$\begin{aligned} P_1^{(1)}(J,n,\varepsilon) &\le C\sum_{J\le j\le \log n} 2^{(1-q/2+q\alpha)j} + C\delta^{q-p}n^{q(1/p-1/2)}\sum_{J\le j\le \log n} 2^{q\alpha j} \\ &\le Cn^{1-q(1/2-\alpha)} + C\delta^{q-p}, \end{aligned}$$

recalling that $1/p-1/2+\alpha=0$. Moreover, as $q>p=(1/2-\alpha)^{-1}$, we get

$$\limsup_{n\to\infty} P_1^{(1)}(J,n,\varepsilon) \le C\delta^{q-p}.$$

This together with (3.10) leads to

$$\limsup_{n\to\infty} P_1(J,n,\varepsilon) \le C\delta^{q-p}.$$

As this last limsup *does not depend on* δ and δ may be choosen arbitrarily small, we conclude that $\limsup_{n\to\infty} P_1(J,n,\varepsilon)=0$, whence Condition 3 of Theorem 3.1 is satisfied. □

3.3. *Long memory.* We now prove Theorem 2.2. For notational simplifications, we assume without loss of generality that $\mathbf{E}\,\epsilon_0^2=1$. Recalling that by Wu and Min (2005, Th.2), $\mathbf{E}\,S_n^2$ is asymptotically equivalent to b_n^2, one can find a constant κ such that for every $n\ge 1$,

$$\sigma_n = (\mathbf{E}\,S_n^2)^{1/2} \le \kappa b_n. \tag{3.13}$$

By the same reference, the square integrability of ϵ_0 is enough to imply the weak-C$[0,1]$ convergence to W^H of $\sigma_n^{-1}\xi_n$ or equivalently of $b_n^{-1}\xi_n$. So, according to the remark after Corollary 3.2, we only need to check (3.1) and Condition 3 of Theorem 3.1 to obtain the weak $\mathrm{H}_\alpha^o[0,1]$ convergence of $b_n^{-1}\xi_n$ to W^H.

Proof of the case $0<\alpha<H-1/2$ *in Theorem 2.2.* The convergence (3.1) follows immediately from Chebyshev inequality:

$$nP\bigl(|X_1|\ge \varepsilon b_n n^{-\alpha}\bigr) \le \frac{n^{2\alpha+1}}{\varepsilon^2 b_n^2}\mathbf{E}\,X_1^2 = O\bigl(n^{2\alpha+1-2H}\ell(n)^{-2}\bigr),$$

since $\alpha<H-1/2$.

Let us keep the same notation $P_1(J,n,\varepsilon)$ as in (3.3) for the probability involved in Condition 3. By stationarity of $(X_i)_{i\in\mathbb{N}}$ and (3.13), we have

$$\mathbf{E}\,\bigl(S_{u_{k+1}}-S_{u_k}\bigr)^2 = \mathbf{E}\,S_{u_{k+1}-u_k}^2 \le \kappa^2 c_\beta^2 (2n2^{-j})^{2H}\ell(2n2^{-j})^2.$$

In view of this estimate, applying Chebyshev inequality leads to

$$\begin{aligned} P_1(J,n,\varepsilon) &\le \frac{4^H\kappa^2}{\varepsilon^2}\sum_{J\le j\le \log n}\frac{\ell(2n2^{-j})^2}{\ell(n)^2}2^{(2\alpha+1-2H)j}\\ &\le \frac{4^H\kappa^2M^2}{\varepsilon^2(1-2^{2\alpha+1-2H})}2^{(2\alpha+1-2H)J}, \end{aligned}$$

noting that $2\alpha+1-2H<0$ and that by slow variation of ℓ

$$M := \sup_{n\ge 1}\frac{\ell(2n)}{\ell(n)} < \infty. \tag{3.14}$$

This entails $\lim_{J\to\infty}\limsup_{n\to\infty} P_1(J,n,\varepsilon)=0$, so the proof of the case $\alpha < H - 1/2$ is complete. □

Proof of the case $H-1/2<\alpha<H$ in Theorem 2.2. To check convergence (3.1), it suffices to show that for any positive ε, $nP(|X_1|\ge \varepsilon n^{H-\alpha}\ell(n)) = o(1)$. By Lemma 3.7 below, the hypothesis (2.10) enables us to write $P(|X_1|\ge t) = t^{-p}g(t)$, with $\lim_{t\to\infty} g(t)=0$. Therefore

$$\begin{aligned} nP\big(|X_1|\ge \varepsilon n^{H-\alpha}\ell(n)\big) &= \varepsilon^{-p}n^{1-p(H-\alpha)}\ell(n)^{-p}g\big(\varepsilon n^{H-\alpha}\ell(n)\big)\\ &= \varepsilon^{-p}\ell(n)^{-p}g\big(\varepsilon n^{H-\alpha}\ell(n)\big) = o(1), \end{aligned}$$

since $p=(H-\alpha)^{-1}$ and $\alpha<H$. So (3.1) is satisfied.

In order to check Condition 3 of Theorem 3, we use the same truncation technics as in the short memory case, with the same level $\Delta_n = \delta n^{1/p}$ but with $1/p = H-\alpha$ instead of $1/2-\alpha$. With obvious adaptations, we also keep the same notations (3.4)–(3.6) and $P_1^{(i)}(J,n,\varepsilon)$. We have again to prove (3.7).

To estimate $P_1^{(2)}(J,n,\varepsilon)$, going back to (3.8), we need some bound for $\mathbf{E}\,|Z_{j,k}^{(2)}|^2$. Write $\widehat{X_k}$, $\widetilde{X_k}$, $\widehat{S_n}$, $\widetilde{S_n}$, for the linear processes obtained by substituting ϵ by $\widehat{\epsilon}$ or $\widetilde{\epsilon}$ respectively and their corresponding partial sums. Then we have

$$Z_{j,k}^{(2)} = \widetilde{S}_{u_{k+1}} - \widetilde{S}_{u_k}$$

whence by stationarity and (3.13),

$$\begin{aligned} \mathbf{E}\,\big|Z_{j,k}^{(2)}\big|^2 = \mathbf{E}\,\widetilde{S}^2_{u_{k+1}-u_k} &\le \kappa^2c_\beta^2(u_{k+1}-u_k)^{2H}\ell^2(u_{k+1}-u_k)\mathbf{E}\,\widetilde{\epsilon}_0^{\,2}\\ &\le 4\kappa^2c_\beta^2n^{2H}2^{-2Hj}\ell^2(2n2^{-j})\mathbf{E}\,\widetilde{\epsilon}_0^{\,2}. \end{aligned}$$

Putting $\gamma := 1+2\alpha-2H$ and pluging the above estimate into (3.13) leads to

$$P_1^{(2)}(J,n,\varepsilon) \le \frac{16\kappa^2}{\varepsilon^2}\mathbf{E}\,\widetilde{\epsilon}_0^{\,2}\sum_{J\le j\le\log n}2^{\gamma j}\frac{\ell^2(2n2^{-j})}{\ell^2(n)} \tag{3.15}$$

$$\le \frac{32M^2\kappa^2}{\varepsilon^2(2^\gamma-1)}\mathbf{E}\,\widetilde{\epsilon}_0^{\,2}n^\gamma. \tag{3.16}$$

Observing that $\gamma = 1+2\alpha-2H = 1-2/p$ and estimating $\mathbf{E}\,\widetilde{\epsilon}_0^{\,2}$ by the inequality (3.23) in Lemma 3.4 provides

$$P_1^{(2)}(J,n,\varepsilon) \le \frac{32M^2\kappa^2p}{\varepsilon^2(2^\gamma-1)(p-2)}\sup_{t\ge\Delta_n}t^pP(|\epsilon|>t).$$

Now from hypothesis (2.10) we get

$$\lim_{n\to\infty} P_1^{(2)}(J,n,\varepsilon) = 0. \tag{3.17}$$

To estimate $P_1^{(1)}(J,n,\varepsilon)$, looking back at (3.11) and (3.12), we see that the only real change is in the control of $\big|\sum_{i=u_k}^{u_{k+1}} \psi_{i-l}\big|^q$. To this end, let us observe that

$$\begin{aligned}
\sup_{k\ge 0}\Bigg|\sum_{i=k-n+1}^{k}\psi_i\Bigg| &= \sup_{k\ge 0}\Bigg|\sum_{1+(k-n)_+<i\le k}\frac{\ell(i)}{i^\beta}\Bigg| \\
&\le \sup_{k\ge 0}\ell(k)\int_{(k-n)_+}^{k}\frac{\mathrm{d}t}{t^\beta} \\
&= \sup_{k\ge 0}\ell(k)\Big(k^{1-\beta}-(k-n)_+^{1-\beta}\Big) \\
&= \sup_{k\ge n}\ell(k)\Big(k^{1-\beta}-(k-n)^{1-\beta}\Big),
\end{aligned}$$

where the last equality relies on the increasingness on $[0,n]$ of the function $t\mapsto \ell(t)\big(t^{1-\beta}-(t-n)_+^{1-\beta}\big)$. Using Lemma 3.6 below leads to

$$\sup_{k\ge 0}\Bigg|\sum_{i=k-n+1}^{k}\psi_i\Bigg| \le cn^{1-\beta}\ell(n),$$

with a constant c depending on β and ℓ. Now we have

$$\begin{aligned}
\Bigg|\sum_{i=u_k}^{u_{k+1}}\psi_{i-l}\Bigg|^q &\le c^{q-2}(u_{k+1}-u_k)^{(q-2)(1-\beta)}\ell^{q-2}(u_{k+1}-u_k)\sigma^2_{u_{k+1}-u_k} \\
&\le 2^q\kappa^2c^{q-2}(n2^{-j})^{q(H-1/2)+1}\ell^q(2n2^{-j}).
\end{aligned}$$

From now on, we denote by C a constant which may depend of ε, q, p, α, c, H, κ and of the distribution of ϵ_0. Its explicit value is allowed to vary from one line to another. Going back to Rosenthal inequality (3.12) with the above estimate and the inequalities (3.22) and (3.25), we get for n large enough:

$$\mathbf{E}\,\big|Z_{j,k}^{(1)}\big|^q \le C\Big(n^{qH}2^{-qHj}\ell^q(2n2^{-j}) + \delta^{q-p}n^{q(H-1/2+1/p)}2^{-(qH-q/2+1)j}\ell^q(2n2^{-j})\Big).$$

Pluging this estimate into (3.11), we obtain

$$\begin{aligned}
P_1^{(1)}(J,n,\varepsilon) \le& \frac{C}{n^{Hq}\ell^q(n)}\sum_{J\le j\le \log n} n^{qH}2^{(1-qH+q\alpha)j}\ell^q(2n2^{-j}) \\
&+\frac{C\delta^{q-p}}{n^{Hq}\ell^q(n)}\sum_{J\le j\le\log n} n^{q(H-1/2+1/p)}2^{q(\alpha-H+1/2)j}\ell^q(2n2^{-j}) \\
\le& C\sum_{J\le j\le\log n}2^{(1-qH+q\alpha)j} + C\delta^{q-p}n^{q(-1/2+1/p)}\sum_{J\le j\le \log n}2^{q(\alpha-H+1/2)j} \\
\le& C2^{(1-qH+q\alpha)J} + C\delta^{q-p}.
\end{aligned}$$

From this bound we get

$$\lim_{J\to\infty}\limsup_{n\to\infty}P_1^{(1)}(J,n,\varepsilon)\le C\delta^{q-p}.$$

Together with (3.17), this gives

$$\lim_{J\to\infty}\limsup_{n\to\infty} P_1(J,n,\varepsilon) \le C\delta^{q-p}.$$

As this last limit *does not depend on* δ and δ may be choosen arbitrarily small, we conclude that $\lim_{J\to\infty}\limsup_{n\to\infty} P_1(J,n,\varepsilon)=0$, whence Condition 3 of Theorem 3.1 is satisfied. □

Proof of the case $\alpha = H-1/2$ in Theorem 2.2. The proof of this special case is obtained by an adaptation of the proof of the case $H-1/2<\alpha<H$. We shall just mention the relevant modifications in the above arguments. Now $p=2$ and we choose as truncation level $\Delta_n = n^{1/2}$. First going back to (3.15), we note that $\gamma=0$, so we have to replace the bound (3.16) by

$$P_1^{(2)}(J,n,\varepsilon) \le \frac{16\kappa^2}{\varepsilon^2}\mathbf{E}\,\widetilde{\epsilon}_0^{\,2}\log n.$$

Under the assumption (2.9), it follows from inequality (3.27) in Lemma 3.5 below that $\mathbf{E}\,\widetilde{\epsilon}_0^{\,2} = o((\log n)^{-1})$, so we get again $\limsup_{n\to\infty} P_1^{(2)}(J,n,\varepsilon)=0$.

Next, choosing $q>3$ and applying (3.27) in Lemma 3.5, the previous estimate of $\mathbf{E}\,|Z_{j,k}^{(1)}|^q$ becomes (with the same convention on the constant C)

$$\mathbf{E}\,\big|Z_{j,k}^{(1)}\big|^q \le C\Big(n^{qH}2^{-qHj}\ell^q(2n2^{-j}) + n^{qH}(\ln n)^{-2}2^{-(qH-q/2+1)j}\ell^q(2n2^{-j})\Big),$$

which leads to

$$\begin{aligned} P_1^{(1)}(J,n,\varepsilon) &\le C\sum_{J\le j\le \log n} 2^{(1-qH+q\alpha)j} + \frac{C}{(\ln n)^2}\sum_{J\le j\le\log n} 2^{-jq(H-1/2-\alpha)} \\ &\le C2^{-J/2} + C\frac{\log n}{(\ln n)^2}. \end{aligned}$$

Hence $\limsup_{n\to\infty} P_1^{(1)}(J,n,\varepsilon) \le C2^{-J/2}$ and $\lim_{J\to\infty}\limsup_{n\to\infty} P_1(J,n,\varepsilon)=0$, which completes the proof. □

3.4. *Miscellaneous technical tools.* We give now a version of Rosenthal inequality for linear processes. Recall first the classical Rosenthal inequality of order $q>2$. It states that for any finite set I of independent random variables Y_i ($i\in I$) such that $\mathbf{E}\,|Y_i|^q<\infty$ (for every $i\in I$), the sum $S_I := \sum_{i\in I} Y_i$ satisfies

$$\mathbf{E}\,\big|S_I\big|^q \le R_q\bigg(\big(\operatorname{Var} S_I\big)^{q/2} + \sum_{i\in I}\mathbf{E}\,|Y_i|^q\bigg), \tag{3.18}$$

where R_q is a universal constant depending only on q.

Lemma 3.3. *Let X be the series*

$$X = \sum_{i=0}^\infty a_i\epsilon_i, \quad \textit{with} \ \sum_{i=0}^\infty a_i^2 < \infty,$$

where the random variables ϵ_i are i.i.d., $\mathbf{E}\,\epsilon_0=0$ and $\mathbf{E}\,|\epsilon_0|^q<\infty$ for some $q>2$. Then the series $\sum_{i=0}^\infty a_i\epsilon_i$ converges in L^q sense and

$$\mathbf{E}\,|X|^q \le R_q\left(\big(\mathbf{E}\,\epsilon_0^2\big)^{q/2}\left(\sum_{i=0}^\infty a_i^2\right)^{q/2} + \mathbf{E}\,|\epsilon_0|^q\sum_{i=0}^\infty |a_i|^q\right), \tag{3.19}$$

where R_q is the universal constant of the Rosenthal inequality (3.18).

Proof. Rosenthal inequality (3.18) applied to the random variables $Y_i = a_i\epsilon_i$ with any non empty subset I of $\mathbb{N}$ reads

$$\mathbf{E}\left|\sum_{i\in I} a_i\epsilon_i\right|^q \le R_q\left(\left(\mathbf{E}\,\epsilon_0^2\right)^{q/2}\left(\sum_{i\in I} a_i^2\right)^{q/2} + \mathbf{E}\,|\epsilon_0|^q\sum_{i\in I}^{\infty}|a_i|^q\right).$$

It follows immediately that the series $\sum_{i=0}^{\infty} a_i\epsilon_i$ fulfils the Cauchy criterion in L^q and hence converges in this space. Now (3.19) follows, taking $I = \{0, 1, \ldots, n\}$ in the above inequality and letting n go to infinity. □

Lemma 3.4. *Let Y be a random variable such that*

$$\Lambda_p(Y) := \sup_{t>0} t^p P(|Y| > t) < \infty \quad \textit{for some } p > 2. \tag{3.20}$$

For any positive T, write

$$\widehat{Y} := Y\mathbf{1}\{|Y| \le T\}, \quad \widetilde{Y} := Y\mathbf{1}\{|Y| > T\}.$$

Write also $\widehat{Y'} := \widehat{Y} - \mathbf{E}\,\widehat{Y}$ and $\widetilde{Y'} := \widetilde{Y} - \mathbf{E}\,\widetilde{Y}$. Then the following estimates are valid with any $q > p$.

$$\mathbf{E}\left|\widehat{Y}\right|^q \le \frac{\Lambda_p(Y)}{q-p}T^{q-p}, \tag{3.21}$$

$$\operatorname{Var}\widehat{Y} \le \mathbf{E}\,Y^2, \tag{3.22}$$

$$\operatorname{Var}\widetilde{Y} \le \frac{p}{p-2}T^{2-p}\sup_{t\ge T} t^p P(|Y| > t). \tag{3.23}$$

If moreover $\mathbf{E}\,Y = 0$, then

$$\left|\mathbf{E}\,\widehat{Y}\right| \le \frac{p}{p-1}T^{1-p}\sup_{t\ge T} t^p P(|Y| > t), \tag{3.24}$$

$$\mathbf{E}\left|\widehat{Y'}\right|^q \le \frac{2^q\Lambda_p(Y)}{q-p}T^{q-p} \quad \textit{for } T \ge T_0, \tag{3.25}$$

where T_0 depends of p, q and of the distribution of Y.

Proof. To check (3.21), write

$$\begin{aligned}
\mathbf{E}\left|\widehat{Y}\right|^q = \int_0^\infty q s^{q-1} P(\left|\widehat{Y}\right| > s)\,\mathrm{d}s &= \int_0^T q s^{q-1} P(\left|\widehat{Y}\right| > s)\,\mathrm{d}s \\
&\le \int_0^T q s^{q-1} P(|Y| > s)\,\mathrm{d}s \\
&\le \sup_{t>0} t^p P(|Y| > t)\int_0^T q s^{q-p-1}\,\mathrm{d}s \\
&= \frac{T^{q-p}}{q-p}\sup_{t>0} t^p P(|Y| > t).
\end{aligned}$$

Next, (3.22) is obvious since $\operatorname{Var}\widehat{Y} \le \mathbf{E}\,\widehat{Y}^2 \le \mathbf{E}\,Y^2$. For (3.23), noting that $P(|\widetilde{Y}| > s) = P(|Y| > \max(s,T))$, we get

$$\begin{aligned}\operatorname{Var}\widetilde{Y} \le \mathbf{E}\,\widetilde{Y}^2 &= \int_0^T 2sP(|Y|>T)\,\mathrm{d}s + \int_T^\infty 2sP(|Y|>s)\,\mathrm{d}s\\ &= T^2P(|Y|>T) + \int_T^\infty 2s^{1-p}s^pP(|Y|>s)\,\mathrm{d}s\\ &\le T^{2-p}\sup_{t\ge T} t^pP(|Y|>t) + \frac{2}{p-2}T^{2-p}\sup_{t\ge T} t^pP(|Y|>t),\end{aligned}$$

which establishes (3.23).

Similarly, if $\mathbf{E}\,Y = 0$, then $\mathbf{E}\,\widehat{Y} = -\mathbf{E}\,\widetilde{Y}$ and we get

$$\begin{aligned}\bigl|\mathbf{E}\,\widehat{Y}\bigr| \le \mathbf{E}\,\bigl|\widetilde{Y}\bigr| &= \int_0^T P(|Y|>T)\,\mathrm{d}s + \int_T^\infty P(|Y|>s)\,\mathrm{d}s\\ &= TP(|Y|>T) + \int_T^\infty s^{-p}\bigl(s^pP(|Y|>s)\bigr)\,\mathrm{d}s\\ &= \left(T^{1-p} + \frac{T^{1-p}}{p-1}\right)\sup_{t\ge T} t^pP(|Y|>t),\end{aligned}$$

which gives (3.24).

By convexity, $\mathbf{E}\,\bigl|\widehat{Y'}\bigr|^q \le 2^{q-1}\bigl(\mathbf{E}\,\bigl|\widehat{Y}\bigr|^q + \bigl|\mathbf{E}\,\widehat{Y}\bigr|^q\bigr)$. By (3.24), $\bigl|\mathbf{E}\,\widehat{Y}\bigr|^q$ goes to 0 when T goes to infinity, whence (3.25) follows. □

Lemma 3.5. *With the notations of lemma 3.4, assume that*

$$\sup_{t>1}(t\ln t)^2P(|Y|>t) < \infty. \tag{3.26}$$

Then with $r(T) := \sup_{t\ge T}(t\ln t)^2P(|Y|>t)$,

$$\operatorname{Var}\widetilde{Y} \le \frac{3r(T)}{\ln T}, \quad \textit{for } T \ge \mathrm{e}. \tag{3.27}$$

If moreover $\mathbf{E}\,Y = 0$, *then for any* $q > 3$,

$$\mathbf{E}\,\bigl|\widehat{Y'}\bigr|^q = O\bigl(T^{q-2}(\ln T)^{-2}\bigr). \tag{3.28}$$

Proof. For every $T > 1$, we can write

$$\begin{aligned}\operatorname{Var}\widetilde{Y} \le \mathbf{E}\,\widetilde{Y}^2 &= T^2P(|Y|>T) + \int_T^\infty 2sP(|Y|>s)\,\mathrm{d}s\\ &\le \frac{r(T)}{(\ln T)^2} + \int_T^\infty \frac{2}{s(\ln s)^2}s^2(\ln s)^2P(|Y|>s)\,\mathrm{d}s\\ &\le \frac{r(T)}{(\ln T)^2} + r(T)\int_T^\infty \frac{2}{s(\ln s)^2}\,\mathrm{d}s\\ &= \left(\frac{1}{(\ln T)^2} + \frac{2}{\ln T}\right)r(T),\end{aligned}$$

whence (3.27) follows.

To check (3.28), we note first that (3.24) remains valid with $p = 2$ and provides the estimate $|\mathbf{E}\,\widehat{Y}|^q = o(T^{-q})$. Hence it is enough to show that $\mathbf{E}\,|\widehat{Y}|^q = O\bigl(T^{q-2}(\ln T)^{-2}\bigr)$. To do that, recall that $\mathbf{E}\,\bigl|\widehat{Y}\bigr|^q \le \int_0^T qs^{q-1}P(|Y|>s)\,\mathrm{d}s$ and split this integral in $\int_0^{T_0} + \int_{T_0}^T$, for $T > T_0$ where $T_0 := \exp(\frac{2}{q-3}) > 1$ is choosen such that $s^{q-3}(\ln s)^{-2}$ increases on $[T_0, \infty)$. This clearly reduces the problem to the following elementary estimation of $\int_{T_0}^{T}$:

$$\int_{T_0}^{T} qs^{q-1} P(|Y| > s)\,\mathrm{d}s = \int_{T_0}^{T} \frac{qs^{q-3}}{(\ln s)^2}(s\ln s)^2 P(|Y| > s)\,\mathrm{d}s$$
$$\leq (T - T_0)\frac{qT^{q-3}}{(\ln T)^2} \sup_{t \geq T_0} (t\ln t)^2 P(|Y| > t).$$

□

Lemma 3.6. *If ℓ is non decreasing and* normalized *slowly varying, then for any $0 < \beta < 1$, there is a constant $C = C(\beta, \ell)$ such that for every $n \geq 1$,*

$$\sup_{k \geq n} \ell(k)\big(k^{1-\beta} - (k-n)^{1-\beta}\big) \leq Cn^{1-\beta}\ell(n). \tag{3.29}$$

Proof. First as $1 - \beta < 1$, we clearly have $k^{1-\beta} \leq (k-n)^{1-\beta} + n^{1-\beta}$ for every $k \geq n$, from which we get

$$\max_{n \leq k \leq 2n} \ell(k)\big(k^{1-\beta} - (k-n)^{1-\beta}\big) \leq \ell(2n)n^{1-\beta}. \tag{3.30}$$

As ℓ is slowly varying, there is a constant $C_1 = C_1(\ell)$ such that $\ell(2n) \leq C_1\ell(n)$.

Next, by concavity of the function $t^{1-\beta}$ on $[0, \infty)$, we have for every $t > n$

$$t^{1-\beta} - (t-n)^{1-\beta} \leq (1-\beta)(t-n)^{-\beta}n. \tag{3.31}$$

Now for every $t \geq 2n$,

$$(t-n)^{-\beta}\ell(t) = t^{-\beta}\ell(t)\Big(\frac{t}{t-n}\Big)^{\beta} \leq 2^{\beta}t^{-\beta}\ell(t).$$

Since ℓ is normalized slowly varying, $t^{-\beta}\ell(t)$ is *ultimately decreasing*, so for large enough n, $t^{-\beta}\ell(t)$ realizes its maximum on $[2n, \infty)$ at $t = 2n$. So going back to (3.31), we can find a constant C_2 depending on β and ℓ such that for every $k > 2n$,

$$\ell(k)\big(t^{1-\beta} - (t-n)^{1-\beta}\big) \leq C_2 n^{1-\beta}\ell(n) \tag{3.32}$$

Now the conclusion follows from (3.30) and (3.32). □

Lemma 3.7. *It holds*

$$\lim_{t\to\infty} t^p P(|X_0| \geq t) = 0 \tag{3.33}$$

if and only if

$$\lim_{t\to\infty} t^p P(|\epsilon_0| \geq t) = 0. \tag{3.34}$$

Proof. To prove the sufficiency of (3.34) for (3.33), let us fix an arbitrary positive δ and define

$$\widehat{\epsilon_j} := \epsilon_j \mathbf{1}\{|\epsilon_j| \leq \delta t\} - \mathbf{E}\,\epsilon_j \mathbf{1}\{|\epsilon_j| \leq \delta t\}, \quad \widetilde{\epsilon_j} := \epsilon_j \mathbf{1}\{|\epsilon_j| > \delta t\} - \mathbf{E}\,\epsilon_j \mathbf{1}\{|\epsilon_j| > \delta t\}.$$

Noting that $\epsilon_j = \widehat{\epsilon_j} + \widetilde{\epsilon_j}$, we have

$$t^p P(|X_0| \geq 2t) \leq t^p P_1 + t^p P_2,$$

where

$$P_1 := P\left(\sum_{j=0}^{\infty} a_j \widehat{\epsilon_j} \geq t\right), \quad P_2 := P\left(\sum_{j=0}^{\infty} a_j \widetilde{\epsilon_j} \geq t\right).$$

To estimate P_2, we apply Chebyshev's inequality combined with inequality (3.23) in Lemma 3.4. Puting $c = \sum_{j=0}^{\infty} a_j^2$ and $c_p = pc/(p-2)$, this gives:

$$P_2 \leq \frac{1}{t^2}\mathbf{E}\left(\sum_{j=0}^{\infty} a_j \widetilde{\epsilon_j}\right)^2 = \frac{c}{t^2}\mathbf{E}\,|\widetilde{\epsilon_0}|^2 \leq c_p \delta^{2-p} t^{-p} \sup_{s \geq \delta t} s^p P(|\epsilon_0| \geq s).$$

To estimate P_1, we combine Markov and Rosenthal inequalities of order $q > p$ with inequalities (3.22) and (3.25) in Lemma 3.4. This gives

$$\begin{aligned} P_1 &\leq t^{-q}\mathbf{E}\left|\sum_{j=0}^{\infty} a_j \widehat{\epsilon}_j\right|^q \\ &\leq R_q t^{-q}\left[\left(\sum_{i=0}^{\infty} |a_i|^2 \mathbf{E}\,|\epsilon_0|^2\right)^{q/2} + \sum_{i=0}^{\infty} |a_i|^q \mathbf{E}\,|\widehat{\epsilon}_0|^q\right] \\ &\leq C t^{-q}\big(1 + \delta^{q-p} t^{q-p}\big) = C\big(t^{-q} + \delta^{q-p} t^{-p}\big), \end{aligned}$$

where the constant C depends on p, q, the sequence (a_i) and the distribution of ϵ_0. Gathering the estimates of P_1 and P_2 gives

$$t^p P(|X_0| > 2t) \leq c_p \delta^{2-p} \sup_{s \geq \delta t} s^p P(|\epsilon_0| \geq s) + C(t^{-q+p} + \delta^{q-p}),$$

whence

$$\limsup_{t \to \infty} t^p P(|X_0| > 2t) \leq C\delta^{q-p}.$$

As δ may be choosen arbitrarily small, as $q > p$ and C does not depend on δ, the sufficiency of (3.34) follows.

Let us prove the necessity of (3.34). We have

$$X_0 = a_0 \epsilon_0 + \sum_{i=1}^{\infty} a_i \epsilon_{-i} = a_0 \epsilon_0 + Z.$$

If $t_0 > 0$ is such that $P(|Z| \leq t_0) \geq 1/2$, we have for $t > t_0$

$$P(|X_0| \geq t) \geq P(|a_0||\epsilon_0| \geq t + t_o) P(|Z| \leq t_0) \geq \frac{1}{2} P(|a_0||\epsilon_0| \geq t + t_0)$$

due to independence of ϵ_0 and Z and the necessity follows. □

References

Yu. A. Davydov. The invariance principle for stationary process. *Theory Probab. Appl.* **15**, 487–498 (1970).

P. Doukhan, G. Oppenheim and M. Taqqu, editors. *Theory and Applications of Long-Range Dependence*. Birkhäuser, Boston-Basel-Berlin (2003).

V.V. Gorodetskii. Convergence to semi-stable Gaussian processes. *Theory Prob. Appl* **22**, 498–508 (1977).

P. Hall and C.C. Heyde. *Martingale Limit Theorem and Its Application.* Academic Press, New York (1980).

D. Hamadouche. Invariance principles in Hölder spaces. *Port. Math.* **57**, 127–151 (2000).

J.R.M. Hosking. Fractional differencing. *Biometrika* **73**, 217–221 (1981).

J.R.M. Hosking. Asymptotic distributions of the sample mean, autocovariances, and autocorrelations of long-memory time series. *J. Econometrics* **73**, 261– 284 (1996).

M. Juodis. Hölderian functional central limit theorem for linear processes. *Liet. Mat. Rink.* **44**, 812–816 (2004).

J. Lamperti. On convergence of stochastic processes. *Trans. Amer. Math. Soc.* **104**, 430–435 (1962).

F. Merlevède, M. Peligrad and S. Utev. Recent advances in invariance principles for stationary sequences. *Probability Surveys* **3**, 1–36 (2006).

M. Peligrad and S. Utev. A new maximal inequality and invariance principle for stationary sequences. *Ann. Probab.* **33** (2), 798–815 (2005).

M. Peligrad and S. Utev. Central limit theorem for stationary linear processes. *Ann. Probab.* **34** (4), 1608–1622 (2006a).

M. Peligrad and S. Utev. Invariance principle for stochastic processes with short memory. In *High Dimensional probability.*, volume 51, pages 18–32. IMS Lecture Notes and Monograph Series (2006b).

P.C.B. Philips and V. Solo. Asymptotics for linear processes. *Ann. Statist.* **20**, 971–1001 (1992).

A. Račkauskas and Ch. Suquet. Invariance principles for adaptive self-normalized partial sums processes. *Stoch. Process. Appl.* **95**, 63–81 (2001).

A. Račkauskas and Ch. Suquet. Hölder norm test statistics for epidemic change. *J. Statist. Plann. Inference* **126** (2), 495–520 (2004a).

A. Račkauskas and Ch. Suquet. Necessary and sufficient condition for the Hölderian functional central limit theorem. *J. Theoret. Probab.* **17** (1), 221–243 (2004b).

A. Račkauskas and Ch. Suquet. Testing epidemic changes in infinite dimensional parameters. *Stat. Inference Stoch. Process.* **9**, 111–134 (2006).

F. Sowell. The fractional unit root distribution. *Econometrica* **58**, 495–505 (1990).

J. H. Stock. Unit roots, structural breaks and trends. In R. F. Engle and D. L. McFadden, editors, *Handbook in econometrics*, volume 4, chapter 46, pages 2739–2841. Elsevier (1994).

W. B. Wu. Unit root for functional of linear processes. *Econom. Theory* **22**, 1–14 (2006).

W. B. Wu and W. Min. On linear processes with dependent innovations. *Stochastic Process. Appl.* **115**, 939–958 (2005).