Laws of large numbers for the occupation time of an age-dependent critical binary branching system

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Abstract. The occupation time of an age-dependent branching particle system in $\mathbb{R}^d$ is considered, where the initial population is a Poisson random field and the particles are subject to symmetric α-stable migration, critical binary branching and random lifetimes. Two regimes of lifetime distributions are considered: lifetimes with finite mean and lifetimes belonging to the normal domain of attraction of a γ-stable law, $\gamma \in (0, 1)$. It is shown that in dimensions $d > \alpha\gamma$ for the heavy-tailed lifetimes case, and $d > \alpha$ for finite mean lifetimes, the occupation time proccess satisfies a strong law of large numbers.

1. Introduction and background

In this paper, we obtain laws of large numbers for the occupation time process of a random population living in the d-dimensional Euclidean space $\mathbb{R}^d$. The evolution of the population is as follows. Any given individual independently develops a spherically symmetric α-stable process during its lifetime τ, where $0 < \alpha \le 2$ and τ is a random variable having a non-arithmetic distribution function, and at the end of its life it either disappears, or is replaced at the site where it died by two newborns, each event occurring with probability $1/2$.

The population starts off from a Poisson random field having Lebesgue measure Λ as its intensity. We postulate the usual independence assumptions in branching systems.

Two regimes for the distribution of τ are considered: either τ has finite mean $\mu > 0$, or τ possesses a distribution function F such that $F(0) = 0$, $F(x) < 1$ for all $x \in [0, \infty)$, and (with $g(u) \sim h(u)$, as $u \to \infty$, meaning $g(u)/h(u) \to \text{const}$, as

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$u \to \infty$)

$$\bar{F}(u) := 1 - F(u) \sim u^{-\gamma}\Gamma(1 - \gamma)^{-1} \quad \text{as} \quad u \longrightarrow \infty, \tag{1.1}$$

where $\gamma \in (0, 1)$ and Γ denotes the Gamma function, i.e., F belongs to the normal domain of attraction of a γ-stable law. In particular, this allows to consider lifetimes with infinite mean.

Let $X(t)$ denote the simple counting measure on $\mathbb{R}^d$ whose atoms are the positions of particles alive at time t, and let $X \equiv \{X(t),\ t \geq 0\}$. When τ has an exponential distribution it is well known that the measure-valued process X is Markov. In the literature there is a lot of work about the Markovian model. Our objective here is to investigate the case when τ is not necessarily an exponential random variable, in which case $\{X(t),\ t \geq 0\}$ is no longer a Markov process. Another striking difference with respect to the case of exponential lifetimes arises when the particle lifetime distribution satisfies (1.1). When the distribution of τ possesses heavy tails, a kind of compensation occurs between longevity of individuals and clumping of the population: heavy-tailed lifetimes enhance the mobility of individuals, favouring in this way the spreading out of particles, and thus counteracting the clumping of the population. Since clumping goes along with local extinction (due to critical branching), a smaller exponent γ suits better for stability of the population. As a matter of fact, Vatutin and Wakolbinger (1999) and Fleischmann et al. (2002) proved that X admits a nontrivial equilibrium distribution if and only if $d \geq \gamma\alpha$. This contrasts with the case of finite-mean (or exponentially distributed) lifetimes, where the necessary and sufficient condition for stability is $d > \alpha$. As we shall see, such qualitative departure from the Markovian model propagates also to other aspects of the branching particle system, such as the large-time behavior of its occupation time.

Recall that the occupation time of the measure-valued process X is again a measure-valued process $J \equiv \{J_t,\ t \geq 0\}$, which is defined by

$$\langle \psi, J_t \rangle := \int_0^t \langle \psi, X_s \rangle ds, \ \ t \geq 0,$$

for all bounded measurable function $\psi : \mathbb{R}^d \to \mathbb{R}_+$, where the notation $\langle \psi, \nu \rangle$ means $\int \psi\, d\nu$.

Limit theorems for occupation times of branching systems have been extensively studied in the context of exponentially distributed lifetimes. Cox and Griffeath (1985) proved a strong law of large numbers for the occupation time of a critical binary branching system. Moreover, in Cox and Griffeath (1985) it is proved a central limit-type theorem for the occupation time of the critical binary branching Brownian motion. Méléard and Roelly (1992) extended the law of large numbers of Cox and Griffeath (1985) to branching populations with general finite-variance critical branching, and quasi-stable particle motions. Bojdecki et al. (2004, 2006a,b) have investigated the limit fluctuations of the re-scaled occupation time $\{J_T(t) := J_{tT}, t \geq 0\}$ of branching systems, T being a parameter which tends to infinity. They have shown that these limits are processes which exhibit long-range dependence behavior, such as fractional Brownian motion and sub-fractional Brownian motion. See also Birkner and Zähle (2007) for related results, where the underlying process is a branching random walk in the d-dimensional lattice.

In this paper we will prove that, in dimensions $d > \alpha\gamma$ for heavy-tailed lifetimes, and $d > \alpha$ for finite-mean lifetimes, the occupation time of the process X satisfies

a strong law of large numbers. Namely, a.s. for any positive continuous function ψ with compact support,

$$t^{-1}\langle \psi, J_t \rangle \longrightarrow \langle \psi, \Lambda \rangle \quad \text{as} \quad t \longrightarrow \infty.$$

Also, we prove that in dimensions $d < \alpha\gamma$ for heavy-tailed lifetimes, and $d < \alpha$ for finite-mean lifetimes, the normalized occupation time $t^{-1}J_t$ converges to zero a.s. in the sense that, with probability 1, for any ball $A \subset \mathbb{R}^d$ of finite radius,

$$t^{-1}\int_0^t 1_{\{X_s(A)>0\}}\, ds \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty. \tag{1.2}$$

These results complement —and partially extend— those of Cox and Griffeath (1985) and Méléard and Roelly (1992). We point out that dimension-dependent behaviors, or parameters, are a typical characteristic in this theory because properties of the branching system treated here are highly related to the transience-recurrence behavior of the particle motions. Notice also that, in contrast with the case of finite-mean lifetimes, in the presence of heavy-tailed lifetimes the dimension restriction varies according to the decay exponent of the tail. This phenomenon is reminiscent of the interplay of population clustering and longevity of individuals quoted above.

Our proofs use techniques from Cox and Griffeath (1985), Iscoe (1986b) and Méléard and Roelly (1992). To prove the strong law of large numbers in case of heavy-tailed lifetimes, we first consider the case of "intermediate dimensions" $\alpha\gamma < d < 2\alpha$, and deal afterward with "large dimensions" $d \geq 2\alpha$. Aiming at applying the Borel-Cantelli lemma, in case of intermediate dimensions we use the re-scaled occupation time process to upper-bound the variance functional of the occupation time. This step employs certain Fourier-transform techniques that we adapted from Bojdecki et al. (2004). We were unable to extend this method to dimensions $d \geq 2\alpha$ due to the lack of proper upper-bounds for the variance functional of the re-scaled occupation time. To deal with the case of large dimensions we follow the approach of Méléard and Roelly (1992). We use a Markovianized branching system, introduced in Section 4.1 below, which allows us to directly apply the well-known self-similarity of the symmetric α-stable transition densities. We remark that, in order to use this procedure, we need to assume that the lifetime distribution possesses a continuous density function. This contrast with the case of low dimensions, where no absolute continuity condition is required. We think, however, that the result should be true for a general lifetime distributions.

In case of a general non-arithmetic lifetime distribution having finite mean, our proof of the law of large numbers is carried out using estimates for the variance functional of the occupation time process, as well as bounds for the α-stable transition densities. The almost sure convergence (1.2) is proved by combining Borel-Cantelli's Lemma with some estimates from Vatutin and Wakolbinger (1999) related to extinction probabilities.

The analysis at the "critical dimensions" $d = \alpha\gamma$ in the heavy-tailed case, and $d = \alpha$ for finite mean lifetimes, is much more difficult to carry out, as can be seen from Fleischmann and Gärtner (1986), where the occupation time (at the critical dimension $d = \alpha/\beta$) of the so-called (d, α, β)-superprocess is considered. The approach there strongly relies on the classical semilinear equation characterizing the Laplace functional of the occupation time, see Lemma 3.4 in Fleischmann and Gärtner (1986). In our case, due to the non-exponential lifetimes, we do not have

the above-mentioned equation. Thus, laws of large numbers for our model at critical dimensions remain to be investigated.

2. Laws of large numbers

Following Bojdecki et al. (2004), we define the re-scaled occupation time process $\{J_T(t) := J_{tT}, \ t \geq 0\}$, i.e., for any positive bounded measurable function φ,

$$\langle \varphi, J_T(t) \rangle := \int_0^{tT} \langle \varphi, X_s \rangle ds = T \int_0^t \langle \varphi, X_{sT} \rangle ds, \ \ t \geq 0. \tag{2.1}$$

Notice that, by Fubini's theorem,

$$\mathbb{E}\langle \varphi, J_T(1) \rangle = \langle \varphi, \Lambda \rangle T, \tag{2.2}$$

since $\mathbb{E}\langle \varphi, X_t \rangle = \langle \varphi, \Lambda \rangle$. We remark that studying the asymptotic behavior of $\langle \varphi, J_t \rangle / t$ as $t \longrightarrow \infty$, is the same as investigating the asymptotic behavior of $\langle \varphi, J_T(1) \rangle / T$ as $T \longrightarrow \infty$.

In what follows, $C_c^+(\mathbb{R}^d)$ denotes the space of nonnegative continuous functions $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ with compact support. The main results of this paper are the following theorems.

Theorem 2.1. *Let F be a non-arithmetic distribution function satisfying (1.1).*
(a) Assume that $\alpha\gamma < d < 2\alpha$. Then, a.s. for any $\varphi \in C_c^+(\mathbb{R}^d)$,

$$T^{-1}\langle \varphi, J_T(1) \rangle \longrightarrow \langle \varphi, \Lambda \rangle \quad as \quad T \longrightarrow \infty. \tag{2.3}$$

(b) Suppose that $d \geq 2\alpha$ and F possesses a continuous density f. Then, a.s. for all $\varphi \in C_c^+(\mathbb{R}^d)$,

$$T^{-1}\langle \varphi, J_T(1) \rangle \longrightarrow \langle \varphi, \Lambda \rangle \quad as \quad T \longrightarrow \infty. \tag{2.4}$$

Our next theorem complements the law of large numbers of Cox and Griffeath (1985) and Méléard and Roelly (1992), which were proved only in the case of exponentially distributed lifetimes.

Theorem 2.2. *Assume that $d > \alpha$, and let F be a non-arithmetic distribution function with finite mean $\mu > 0$. Then, a.s. for any $\varphi \in C_c^+(\mathbb{R}^d)$,*

$$T^{-1}\langle \varphi, J_T(1) \rangle \longrightarrow \langle \varphi, \Lambda \rangle \quad as \quad T \longrightarrow \infty. \tag{2.5}$$

Theorem 2.3. *Let F be a non-arithmetic distribution function which satisfies (1.1), and assume that $d < \alpha\gamma$. Then, a.s. for any ball $A \subset \mathbb{R}^d$ of finite radius,*

$$T^{-1} \int_0^T 1_{\{X_s(A) > 0\}} \, ds \longrightarrow 0 \quad as \quad T \longrightarrow \infty. \tag{2.6}$$

Remark 2.4. (a) Notice that condition $\alpha\gamma < d < 2\alpha$ in Theorem 2.1 allows $d \leq \alpha$, which contrasts with the classical case of exponentially distributed lifetimes, where $d > \alpha$.
(b) When the particle lifetimes have an exponential distribution with mean λ^{-1}, Theorem 2.2 reduces to Theorem 4 of Méléard and Roelly (1992).
(c) In case of low dimensions $d < \alpha\gamma$, a genuine counterpart to Theorem 2.1 would be a statement ensuring a.s. vague convergence of $T^{-1}J_T(1)$ to the zero measure as $T \to \infty$. This was proved by Sawyer and Fleischmann (1979) for the occupation time of critical branching Brownian motion (see also Iscoe, 1986b for a related result regarding super-Brownian motion's occupation time). For our model, here

we prove only the slightly weaker result (2.6), which implies that, with probability 1, the proportion of time that the branching system charges any given bounded set vanishes asymptotically as $T \to \infty$. The specific form (2.6) of a.s. convergence was suggested to us by an anonymous referee.

(d) The extent of dimensions in our results is narrow due to our choice of critical, binary-branching mechanism. A less restrictive assumption, such as critical $(1+\beta)$-branching, $\beta \in (0,1)$, would expand the dimension range.

3. Some moment calculations

Let $Z_t(A)$ denote the number of individuals living in $A \in \mathcal{B}(\mathbb{R}^d)$ at time t, in a population starting with one particle at time $t = 0$. Following Kaj and Sagitov (1998) we define

$$Q_t\varphi(x) := \mathbb{E}_x\left[1 - e^{-\langle\varphi,Z_t\rangle}\right], \quad x \in \mathbb{R}^d, \quad t \geq 0, \tag{3.1}$$

where $\varphi \in C_c^+(\mathbb{R}^d)$ and $\mathbb{E}_x$ means that the initial particle is located at $x \in \mathbb{R}^d$. Since the initial population X_0 is
Poissonian, we have

$$\begin{aligned}
\mathbb{E}e^{-\langle\varphi,X_t\rangle} &= \exp\left(-\int \mathbb{E}_x\left[1 - e^{-\langle\varphi,Z_t\rangle}\right]dx\right) \\
&= \exp\left(-\int Q_t\varphi(x)dx\right), \quad \varphi \in C_c^+(\mathbb{R}^d). \tag{3.2}
\end{aligned}$$

Let $\{\tau_k, k \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function F, and let

$$N_t = \sum_{k=1}^{\infty} 1_{\{S_k \leq t\}}, \quad t \geq 0,$$

where the random sequence $\{S_k, k \geq 0\}$ is recursively defined by

$$S_0 = 0, \quad S_{k+1} = S_k + \tau_k, \quad k \geq 0.$$

For any $p = 1, 2, \ldots,$ $0 < t_p \leq t_{p-1}, \ldots, t_1 < \infty$, $\varphi_1, \varphi_2, \ldots, \varphi_p \in C_c^+(\mathbb{R}^d)$ and $\theta_1, \ldots, \theta_p \in \mathbb{R}$, we define $\bar{t} = (t_1, t_2, \ldots, t_p)$, $\bar{t} - s = (t_1 - s, t_2 - s, \ldots, t_p - s)$, $\theta_{(p)} = (\theta_1, \ldots, \theta_p)'$ and

$$Q_{\bar{t}}^p\theta_{(p)}(x) = \mathbb{E}_x\left[1 - e^{-\sum_{j=1}^p \theta_j\langle\varphi_j, Z_{t_j}\rangle}\right].$$

Let $\{B_s, s \geq 0\}$ denote the spherically symmetric α-stable process in $\mathbb{R}^d$, with transition density functions $\{p_t(x, y), t > 0, x, y \in \mathbb{R}^d\}$, and semigroup $\{\mathcal{S}_t, t \geq 0\}$. Our moment calculations use the following result which is borrowed from Kaj and Sagitov (1998, Section 4.3), and which we include for convenience.

Proposition 3.1. *The function $Q_{\bar{t}}^p\theta_{(p)}$ satisfies*

$$\begin{aligned}
Q_{\bar{t}}^p\theta_{(p)}(x) &= \mathbb{E}_x\left[1 - e^{-\sum_{j=1}^p \theta_j\varphi_j(B_{t_j})} - \int_0^{t_p} \frac{1}{2}\left(Q_{\bar{t}-s}^p\theta_{(p)}(B_s)\right)^2 dN_s \right.\\
&\left. \quad - \sum_{i=1}^{p-1}\left(1 - e^{-\sum_{j=i+1}^p \theta_j\varphi_j(B_{t_j})}\right)\int_{t_{i+1}}^{t_i} \frac{1}{2}\left(Q_{\bar{t}-s}^i\theta_{(i)}(B_s)\right)^2 dN_s\right].
\end{aligned}$$

As in (3.2), since the initial population is Poissonian we have

$$
\mathbb{E}\left[e^{-\sum_{j=1}^{p}\theta_j\langle\varphi_j,X_{t_j}\rangle}\right] \;=\; \exp\left(-\int \mathbb{E}_x\left[1-e^{-\sum_{j=1}^{p}\theta_j\langle\varphi,Z_{t_j}\rangle}\right]dx\right)
$$

$$
=\; \exp\left(-\int Q_t^p\theta_{(p)}(x)dx\right). \tag{3.3}
$$

Using criticality of the branching, and that Lebesgue measure is invariant for the semigroup of the symmetric α-stable process, it is easy to see that

$$
m(t,\varphi) := \mathbb{E}[\langle\varphi,X_t\rangle] = \langle\varphi,\Lambda\rangle, \quad t\geq 0, \quad \varphi\in C_c^+(\mathbb{R}^d). \tag{3.4}
$$

Lemma 3.2. *Let $0 < s \leq t < \infty$ and $\psi,\varphi\in C_c^+(\mathbb{R}^d)$. Then,*

$$
C_x(s,\varphi;t,\psi) \;:=\; \mathbb{E}_x\left[\langle\varphi,Z_s\rangle\langle\psi,Z_t\rangle\right]
$$

$$
=\; \mathbb{E}_x\left[\varphi(B_s)\psi(B_t)+\int_0^s m_{B_r}(t-r,\psi)m_{B_r}(s-r,\varphi)dN_r\right], \tag{3.5}
$$

where $m_x(t,\varphi) = \mathbb{E}_x[\langle\varphi,Z_t\rangle]$.

Proof: In order to preserve the notation in Proposition 3.1, we put $p=2$, $t_1=t$, $t_2=s$, $\varphi_1=\psi$ and $\varphi_2=\varphi$. Then we have

$$
C_x(t_1,\varphi_1;t_2,\varphi_2) = -\frac{\partial^2}{\partial\theta_1\partial\theta_2}Q_t^2\theta_{(2)}(x)\Big|_{\theta_1=\theta_2=0^+},
$$

where

$$
\frac{\partial^2}{\partial\theta_1\partial\theta_2}Q_t^2\theta_{(2)}(x)
$$

$$
=\; \mathbb{E}_x\Bigg[-\varphi_1(B_{t_1})\varphi_2(B_{t_2})e^{-\theta_1\varphi(B_{t_1})-\theta_2\varphi_2(B_{t_2})}
$$

$$
-\int_0^{t_2}\frac{\partial}{\partial\theta_2}Q_{t-r}^2\theta_{(2)}(B_r)\frac{\partial}{\partial\theta_1}Q_{t-r}^2\theta_{(2)}(B_r)dN_r
$$

$$
-\int_0^{t_2}\left(Q_{t-r}^2\theta_{(2)}(B_r)\right)\frac{\partial^2}{\partial\theta_2\partial\theta_1}Q_{t-r}^2\theta_{(2)}(B_r)dN_r
$$

$$
-\varphi_2(B_{t_2})e^{-\theta\varphi_2(B_{t_2})}\int_{t_1}^{t_2}\left(Q_{t_2-r}^1\theta_1(B_r)\right)\frac{\partial}{\partial\theta_1}Q_{t_2-r}^1\theta_1(B_r)dN_r\Bigg].
$$

Evaluating at $\theta_1=\theta_2=0$ we finish the proof. $\qquad\square$

Proposition 3.3. *Let $0 < s \leq t < \infty$ and $\psi,\varphi\in C_c^+(\mathbb{R}^d)$. Then,*

$$
C(s,\varphi;t,\psi) \;:=\; \mathrm{Cov}\left(\langle\varphi,X_s\rangle,\langle\psi,X_t\rangle\right)
$$

$$
=\; \langle\varphi\mathcal{S}_{t-s}\psi,\Lambda\rangle+\int_0^s\langle(\mathcal{S}_{s-r}\varphi)(\mathcal{S}_{t-r}\psi),\Lambda\rangle\, dU(r), \tag{3.6}
$$

where $U(r) = \sum_{k=0}^\infty F^{*k}(r)$.

Proof: We put $p = 2$ in (3.3) and use the same notations as in the proof of Lemma 3.2. Then,

$$
\begin{aligned}
&\mathbb{E}\left[\langle\varphi_1, X_{t_1}\rangle\langle\varphi_2, X_{t_2}\rangle\right] \\
&= \left.\frac{\partial^2}{\partial\theta_1\partial\theta_2}\exp\left(-\int Q_{\bar{t}}^2\theta_{(2)}(x)\,dx\right)\right|_{\theta_1=\theta_2=0^+} \\
&= \left[-\frac{\partial^2}{\partial\theta_1\partial\theta_2}\int Q_{\bar{t}}^2\theta_{(2)}(x)dx \right. \\
&\quad \left.\left.+ \int\frac{\partial}{\partial\theta_1}Q_{\bar{t}}^2\theta_{(2)}(x)\,dx\int\frac{\partial}{\partial\theta_2}Q_{\bar{t}}^2\theta_{(2)}(x)\,dx\right]\right|_{\theta_1=\theta_2=0^+} \\
&= \int C_x(t_1,\varphi_1;t_2,\varphi_2)dx + \int m_x(t_1,\varphi_1)dx\int m_x(t_2,\varphi_2)dx,
\end{aligned}
$$

and from Lemma 3.2 we obtain

$$
C(s,\varphi;t,\psi) = \int_{\mathbb{R}^d}\mathbb{E}_x\left[\varphi(B_s)\psi(B_t) + \int_0^s m_{B_r}(t-r,\psi)m_{B_r}(s-r,\varphi)dN_r\right]dx,
\tag{3.7}
$$

which completes the proof. $\square$

4. Markovianizing an age-dependent branching system

In this section we introduce a Markovian branching system $\{\bar{X}_t,\ t \geq 0\}$ which will be used to prove Theorem 2.1 (b). Let $X \equiv \{X_t,\ t \geq 0\}$ be the branching system defined in Section 1. For any $t \geq 0$, let $\bar{X}_t$ denote the population in $\mathbb{R}_+ \times \mathbb{R}^d$ ($\mathbb{R}_+ = [0,\infty)$) obtained by attaching to each individual $\delta_x \in X_t$ its age. Namely, for each $t \geq 0$,

$$
\bar{X}_t = \sum_i \delta_{(\eta_t^i, \xi_t^i)},
\tag{4.1}
$$

where η_t^i and ξ_t^i denotes respectively, the age and position of the ith particle at time t, and the summation is over all particles alive at time t. Let us assume that $\bar{X}_0$ is a Poisson random field on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $F \times \Lambda$. Here, F also means the Lebesgue-Stieltjes measure corresponding to F. The probability generating function of the branching law is denoted by Φ. Thus, for critical binary branching, $\Phi(s) \equiv \frac{1}{2}(1+s^2)$, $-1 \leq s \leq 1$.

Given a counting measure ν on $\mathbb{R}_+ \times \mathbb{R}^d$, and a measurable function $\phi : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow (0,1]$, we define

$$
G_\phi(\nu) := \exp\left(\langle\log\phi,\nu\rangle\right).
$$

It can be shown that the infinitesimal generator of $\{\bar{X}_t,\ t \geq 0\}$ evaluated at the function $G_\phi(\nu)$ is given by

$$
\mathcal{G}G_\phi(\nu) = G_\phi(\nu)\left\langle\frac{\mathcal{L}\phi(*,\cdot) + \lambda(*)[\Phi(\phi(0,\cdot)) - \phi(0,\cdot)]}{\phi(*,\cdot)},\nu\right\rangle,
\tag{4.2}
$$

where

$$
\lambda(u) = \frac{f(u)}{1 - F(u)},\quad u \geq 0,
\tag{4.3}
$$

is the hazard rate function associated to F, and

$$
\mathcal{L}\phi(u,x) = \frac{\partial\phi(u,x)}{\partial u} + \Delta_\alpha\phi(u,x) - \lambda(u)\left[\phi(u,x) - \phi(0,x)\right],
\tag{4.4}
$$

where the function ϕ is such that $\phi(\cdot, x) \in C_b^1(\mathbb{R}_+)$ for any $x \in \mathbb{R}^d$, and $\phi(u, \cdot) \in C_c^\infty(\mathbb{R}^d)$ for any $u \in \mathbb{R}_+$. Here $C_b^1(\mathbb{R}_+)$ denotes the set of all bounded functions with continuous first derivative, and $C_c^\infty(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$, having compact support. The operator $\mathcal{L}$ is the infinitesimal generator of a Markov process on $\mathbb{R}_+ \times \mathbb{R}^d$ whose semigroup is denoted by $\{\tilde{T}_t, t \geq 0\}$, see Murillo-Salas (2008) for details.

Proposition 4.1. *Let* $\bar{X} \equiv \{\bar{X}_t, t \geq 0\}$ *as before and let* $\bar{X}_0$ *be a Poisson random field on* $\mathbb{R}_+ \times \mathbb{R}^d$ *with intensity measure* $F \times \Lambda$. *The joint Laplace functional of the branching particle system* $\bar{X}$ *and its occupation time is given by*

$$\mathbb{E}\left[e^{-\langle \psi, \bar{X}_t \rangle - \int_0^t \langle \phi, \bar{X}_s \rangle ds}\right] = e^{-\langle V_t^\psi \phi, F \times \Lambda \rangle}, \quad t \geq 0,$$

for all measurable functions $\psi, \phi : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$ *with compact support, where* $V_t^\psi \phi$ *satisfies, in the mild sense, the non-linear evolution equation*

$$\begin{aligned}
\frac{\partial}{\partial t} V_t^\psi \phi(u, x) &= \mathcal{L} V_t^\psi \phi(u, x) - \lambda(u)[\Phi(1 - V_t^\psi \phi(0, x)) - (1 - V_t^\psi \phi(0, x))] \\
&\quad + \phi(u, x)(1 - V_t^\psi \phi(u, x)), \\
V_0^\psi \phi(u, x) &= 1 - e^{-\psi(u, x)}.
\end{aligned} \tag{4.5}$$

Proof: The proof is carried out using the martingale problem for $\{\bar{X}_t, t \geq 0\}$, and Itô's formula. We omit the details. $\square$

5. Proof of Theorem 2.1

We shall prove the law of large numbers in two steps. First we show that the result holds for intermediate dimensions $\alpha\gamma < d < 2\alpha$; this part of the proof relies on the non-Markovian branching system defined in Section 1, and uses upper bounds for the covariance functional. In the second step, we consider "large" dimensions $d \geq 2\alpha$, and in this case we use the Markovianized branching system described in Section 4.

5.1. *Proof of Theorem 2.1 (a).* In this section we assume that $\alpha\gamma < d < 2\alpha$.

Lemma 5.1. *Suppose that the hypothesis in Theorem 2.1 hold. Then, for each* $\epsilon > 0$ *and all* $T > 0$ *large enough,*

$$P\left(\left|T^{-1}\langle \varphi, J_T(1) \rangle - \langle \varphi, \Lambda \rangle\right| > \epsilon\right) \leq \frac{2}{\epsilon^2}\left(c_3 T^{-2} + c_1 T^{-1} + c_2 T^{-d/\alpha} + c_4 T^{\gamma - d/\alpha}\right),$$

for some positive constants $c_1, \ldots, c_4$.

Proof: Let $\epsilon > 0$ be given. Then, using Chebyshev's inequality and (2.2),

$$\begin{aligned}
P\left(\left|T^{-1}\langle \varphi, J_T(1) \rangle - \langle \varphi, \Lambda \rangle\right| > \epsilon\right) &\leq \frac{1}{\epsilon^2}\mathbb{E}\left(T^{-1}\langle \varphi, J_T(1) \rangle - \langle \varphi, \Lambda \rangle\right)^2 \\
&= \frac{1}{\epsilon^2 T^2}\text{Cov}\left(\langle \varphi, J_T(1) \rangle, \langle \varphi, J_T(1) \rangle\right) \\
&= \frac{1}{\epsilon^2}\int_0^1 \int_0^1 \text{Cov}\left(\langle \varphi, X_{sT} \rangle, \langle \varphi, X_{tT} \rangle\right) dt\, ds,
\end{aligned}$$

where the last equality follows from (2.1). By changing the order of integration we obtain that

$$P\left(\left|T^{-1}\langle\varphi, J_T(1)\rangle - \langle\varphi, \Lambda\rangle\right| > \epsilon\right) \le \frac{2}{\epsilon^2}\int_0^1 dv\int_0^v \mathrm{Cov}\left(\langle\varphi, X_{uT}\rangle, \langle\varphi, X_{vT}\rangle\right) du.$$
(5.1)

Therefore, from Proposition 3.3 we deduce that

$$P\left(\left|T^{-1}\langle\varphi, J_T(1)\rangle - \langle\varphi, \Lambda\rangle\right| > \epsilon\right) \le (I) + (II),$$
(5.2)

with

$$(I) := \frac{2}{\epsilon^2}\int_0^1 dv\int_0^v du\langle\varphi\mathcal{S}_{T(v-u)}\varphi, \Lambda\rangle$$

and

$$(II) := \frac{2}{\epsilon^2}\int_0^1 dv\int_0^v du\int_0^u dU(Tr)\langle\varphi\mathcal{S}_{T(v+u-2r)}\varphi, \Lambda\rangle,$$

where, to obtain (II), we used self-adjointness of $\mathcal{S}_t$ with respect to Λ, $t \ge 0$. Our next goal is to derive useful upper bounds for the two integrals (I) and (II). Firstly, by performing the change of variables $s = (v-u)T$ and $t = vT$, we get that

$$\frac{\epsilon^2}{2}(I) = \frac{1}{T^2}\int_0^T dt\int_0^t ds\langle\varphi\mathcal{S}_s\varphi, \Lambda\rangle$$

$$= \frac{1}{T^2}\int_0^A dt\int_0^t ds\langle\varphi\mathcal{S}_s\varphi, \Lambda\rangle + \frac{1}{T^2}\int_A^T dt\int_0^t ds\langle\varphi\mathcal{S}_s\varphi, \Lambda\rangle$$

for any $A > 0$, where

$$\int_0^t\langle\varphi\mathcal{S}_s\varphi, \Lambda\rangle ds = \int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)p_s(x-y)\varphi(y)dy\,dx\,ds$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)\varphi(y)\int_0^t p_s(x-y)ds\,dy\,dx$$

$$\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)\varphi(y)\mathrm{const.}\left(|x-y|^{\alpha-d} + t^{1-d/\alpha}\right)dy\,dx$$

since

$$\int_0^t p_s(x-y)ds \le \mathrm{const.}\left(|x-y|^{\alpha-d} + t^{1-d/\alpha}\right)$$

due to self-similarity of the α-stable transition densities. Notice that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)\varphi(y)|x-y|^{\alpha-d}dy\,dx < \infty,$$

which, for $d > \alpha$, follows from Lemma 5.3 in Iscoe (1986b). Hence,

$$\int_A^T dt\int_0^t\langle\varphi\mathcal{S}_s\varphi, \Lambda\rangle\,ds$$

$$\le \mathrm{const.}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)\varphi(y)\int_A^T\left(|x-y|^{\alpha-d} + t^{1-d/\alpha}\right)dt\,dy\,dx$$

$$= c_1(T-A) + c_2(T^{1-d/\alpha} - A^{1-d/\alpha})$$

for some constants $c_1, c_2 > 0$. Therefore,

$$(I) \le \frac{2}{\epsilon^2}\left(\frac{c_3}{T^2} + c_1\frac{T}{T^2} + c_2\frac{T^{2-d/\alpha}}{T^2}\right),$$
(5.3)

where

$$c_3 = \int_0^A dt \int_0^t \langle \varphi \mathcal{S}_s \varphi, \Lambda \rangle ds.$$

Before estimating the integral (II), we recall that $U(t) \sim t^\gamma / \Gamma(1+\gamma)$ as $t \to \infty$ because of $\bar{F}(t) \sim t^{-\gamma} / \Gamma(1-\gamma)$, see Bingham et al. (1989, p. 361). Then, writing $\hat{\varphi}$ for the Fourier transform of φ, we obtain

$$
\begin{aligned}
\frac{\epsilon^2}{2}(II) &= \int_0^1 dv \int_0^v du \int_0^u dU(Tr) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dy |\hat{\varphi}(y)|^2 e^{-T(v+u-2r)|y|^\alpha} \\
&= \frac{1}{(2\pi)^d} \int_0^1 dv \int_0^v du \int_{\mathbb{R}^d} dy |\hat{\varphi}(y)|^2 \int_0^u dU(Tr) e^{-T(v+u-2r)|y|^\alpha} \\
&\sim \frac{\gamma T^\gamma}{\Gamma(1+\gamma)(2\pi)^d} \int_0^1 dv \int_0^v du \int_{\mathbb{R}^d} dy |\hat{\varphi}(y)|^2 \int_0^u e^{-T(v+u-2r)|y|^\alpha} r^{\gamma-1} dr,
\end{aligned}
$$

and, after the change of variables $z = (T(v+u-2r))^{1/\alpha} y$, we conclude that

$$
\begin{aligned}
\frac{\epsilon^2}{2}(II) &\sim \frac{\gamma T^\gamma}{\Gamma(1+\gamma)(2\pi)^d} \int_0^1 dv \int_0^v du \int_{\mathbb{R}^d} dz \int_0^u T^{-d/\alpha}(v+u-2r)^{-d/\alpha} \\
&\quad \times |\hat{\varphi}(T^{-d/\alpha}(v+u-2r)^{-d/\alpha} z)|^2 e^{-|z|^\alpha} r^{\gamma-1} dr \\
&\leq \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \Lambda \rangle^2}{\Gamma(1+\gamma)(2\pi)^d} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 dv \int_0^v du \int_0^u (v+u-2r)^{-d/\alpha} r^{\gamma-1} dr,
\end{aligned}
$$

where to obtain the last inequality we have used the well known fact that $|\hat{\varphi}(z)| \leq (2\pi)^{-d} \langle |\varphi|, \Lambda \rangle$ for any L^1-function φ.

Changing the order of integration into the above expression yields

$$
\begin{aligned}
&= \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \Lambda \rangle^2}{\Gamma(1+\gamma)(2\pi)^d} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 dv \int_0^v r^{\gamma-1} \int_r^v du (u+v-2r)^{-d/\alpha} dr \\
&= \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \Lambda \rangle^2}{\Gamma(1+\gamma)(2\pi)^d} \\
&\quad \cdot \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 dv \int_0^v r^{\gamma-1} \frac{2^{1-d/\alpha}(v-r)^{1-d/\alpha} - (v-r)^{1-d/\alpha}}{1-d/\alpha} dr \\
&= \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \Lambda \rangle^2}{\Gamma(1+\gamma)(2\pi)^d} \frac{2^{1-d/\alpha}-1}{1-d/\alpha} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 dv \int_0^v r^{\gamma-1}(v-r)^{1-d/\alpha} dr.
\end{aligned}
$$

Notice that in the above calculations we have implicitly assumed that $d \neq \alpha$. The case $d = \alpha$ can be treated in a similar way (and renders the same conclusion (5.4)). Changing again the order of integration we get

$$
\begin{aligned}
&= \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \Lambda \rangle^2}{\Gamma(1+\gamma)(2\pi)^d} \frac{2^{1-d/\alpha}-1}{1-d/\alpha} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 r^{\gamma-1} \int_r^1 dv(v-r)^{1-d/\alpha} dr \\
&= \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \Lambda \rangle^2}{\Gamma(1+\gamma)(2\pi)^d} \frac{2^{1-d/\alpha}-1}{(1-d/\alpha)(2-d/\alpha)} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 r^{\gamma-1}(1-r)^{2-d/\alpha} dr,
\end{aligned}
$$

where the last equality is finite since by assumption $d < 2\alpha$. Hence, for T large enough

$$(II) \leq \frac{2}{\epsilon^2} c_4 T^{\gamma-d/\alpha}. \tag{5.4}$$

Therefore,

$$P\left(|T^{-1}\langle\varphi, J_T(1)\rangle - \langle\varphi, \Lambda\rangle| > \epsilon\right) \leq \frac{2}{\epsilon^2}\left(c_3 T^{-2} + c_1 T^{-1} + c_2 T^{-d/\alpha} + c_4 T^{\gamma - d/\alpha}\right).$$

$\square$

Proof of Theorem 2.1 (a): Let $\epsilon > 0$ and $a > 1$ be given constants, and let $T_n = a^n$ for $n = 1, 2, \ldots$. Then,

$$\sum_{n=1}^{\infty} P\left(|T_n^{-1}\langle\varphi, J_{T_n}(1)\rangle - \langle\varphi, \Lambda\rangle| > \epsilon\right)$$

$$\leq \frac{2}{\epsilon^2}\sum_{n=1}^{\infty}\left(c_3 T_n^{-2} + c_1 T_n^{-1} + c_2 T_n^{-d/\alpha} + c_4 T_n^{\gamma - d/\alpha}\right) < \infty$$

due to the assumption $d > \gamma\alpha$. Therefore, for any given $\varphi \in C_c^+(\mathbb{R}^d)$, a.s.,

$$T_n^{-1}\langle\varphi, J_{T_n}(1)\rangle \longrightarrow \langle\varphi, \Lambda\rangle \quad \text{as} \quad n \longrightarrow \infty.$$

Now we observe that, for each $T > 1$, there exists some non-negative integer $n(T)$ such that $a^{n(T)} \leq T \leq a^{n(T)+1}$, and that $n(T) \longrightarrow \infty$ as $T \longrightarrow \infty$. Hence,

$$\frac{\langle\varphi, J_{a^{n(T)}}(1)\rangle}{a^{n(T)+1}} \leq \frac{\langle\varphi, J_T(1)\rangle}{T} \leq \frac{\langle\varphi, J_{a^{n(T)+1}}(1)\rangle}{a^{n(T)}},$$

and

$$\frac{\langle\varphi, \Lambda\rangle}{a} \leq \liminf_{T\to\infty}\frac{\langle\varphi, J_T(1)\rangle}{T} \leq \limsup_{T\to\infty}\frac{\langle\varphi, J_T(1)\rangle}{T} \leq \langle\varphi, \Lambda\rangle a,$$

these inequalities being true for any $a > 1$. Letting $a \to 1$ yields that

$$\lim_{T\longrightarrow\infty} T^{-1}\langle\varphi, J_t(1)\rangle = \langle\varphi, \Lambda\rangle$$

a.s., where the null set (where the limit may not exist) depends upon φ. Nonetheless, a null set can be chosen not to depend on φ as is the proof of Theorem 1 in Iscoe (1986a). $\square$

5.2. *Proof of Theorem 2.1 (b).* Throughout this section we assume that $d \geq 2\alpha$. The proof of part (b) in Theorem 2.1 follows, as in part (a), from Chebyshev's inequality

$$P\left\{\frac{|\langle\phi, J_t\rangle - \langle\phi, \Lambda\rangle|}{t} > \epsilon\right\} \leq \frac{1}{t^2\epsilon^2}\text{Var}\langle\phi, J_t\rangle, \quad t \geq 0, \quad \epsilon > 0 \tag{5.5}$$

and Lemma 5.2 below. Recall that λ is defined in (4.3).

Lemma 5.2. *i) Let $\phi : \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function with compact support. Then, for each $t \geq 0$,*

$$\mathbb{E}\langle\phi, J_t\rangle = \langle\phi, \Lambda\rangle t, \tag{5.6}$$

and

$$\text{Var}\langle\phi, J_t\rangle \leq \langle\lambda, F\rangle\text{Const}(\phi)(t + t^{3-d/\alpha}) + 2\text{Const}(\phi)(t + t^{2-d/\alpha}). \tag{5.7}$$

Proof: First we prove (5.6). For the given function ϕ we define the extended function $\bar\phi(u, x) \equiv \phi(x)$, $(u, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Then, for any $k \geq 0$ we define

$$L_t(k\bar\phi) = \mathbb{E}\left[e^{-k\int_0^t\langle\bar\phi, \bar X_s\rangle ds}\right] = e^{-\langle V_t(k\bar\phi), F\times\Lambda\rangle},$$

where $V_t(k\bar\phi)$ satisfies (4.5) with $\bar\phi$ substituted by $k\bar\phi$, and $\psi \equiv 0$. Notice that

$$
\begin{aligned}
\mathbb{E}\langle \phi, J_t \rangle &= -\frac{d}{dk}\mathbb{E}\left[\exp -\langle k\bar\phi, J_t\rangle\right]|_{k=0^+} \\
&= \left\langle \frac{d}{dk}V_t^0(k\bar\phi), F \times \Lambda \right\rangle \exp\left(-\langle V_t^0(k\bar\phi), F \times \Lambda\rangle\right)\bigg|_{k=0^+}.
\end{aligned}
$$

Thus, putting $\dot{V}_t\bar\phi := \frac{d}{dk}V_t^0(k\bar\phi)|_{k=0^+}$ and recalling that $V_t^0(0\bar\phi) = 0$, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\dot{V}_t\bar\phi(u, x) &= \mathcal{L}\dot{V}_t\bar\phi(u, x) + \bar\phi(u, x) \\
\dot{V}_0\bar\phi(u, x) &= 0,
\end{aligned}
$$

or

$$
\dot{V}_t\bar\phi(u, x) = \int_0^t \tilde{T}_{t-s}\bar\phi(u, x)ds = \int_0^t \mathcal{S}_{t-s}\phi(x)ds. \tag{5.8}
$$

Consequently, using that Λ is invariant for the α-stable semigroup,

$$
\mathbb{E}\langle \bar\phi, J_t \rangle = \langle \dot{V}_t\bar\phi, F \times \Lambda \rangle = \left\langle \int_0^t \mathcal{S}_{t-s}\phi ds, \Lambda \right\rangle = \langle \phi, \Lambda \rangle t.
$$

This proves (5.6).

The proof of (5.7) goes as follows. Define $\bar\phi$ as before. Differentiating $V_t(k\bar\phi)$ with respect to k and using equation (4.5), we obtain

$$
\begin{aligned}
&\frac{\partial^2}{\partial t\partial k}V_t(k\bar\phi)(u, x) \\
&= \mathcal{L}\frac{\partial}{\partial k}V_t(k\bar\phi)(u, x) + \bar\phi(u, x)(1 - V_t(k\bar\phi)(0, x)) - k\bar\phi(u, x)\frac{\partial}{\partial k}V_t(k\bar\phi)(u, x) \\
&\quad -\lambda(u)\left[-\Phi'(1 - V_t(k\bar\phi(0, x))\frac{\partial}{\partial k}V_t(k\bar\phi)(0, x) + \frac{\partial}{\partial k}V_t(k\bar\phi)(0, x)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial^3}{\partial t\partial k^2}V_t(k\bar\phi)(u, x) \\
&= \mathcal{L}\frac{\partial^2}{\partial k^2}V_t(k\bar\phi)(u, x) - 2\bar\phi(u, x)\frac{\partial}{\partial k}V_t(k\bar\phi)(u, x) - k\bar\phi(u, x)\frac{\partial^2}{\partial k^2}V_t(k\bar\phi)(u, x) \\
&\quad -\lambda(u)\left[\Phi''\left(1 - V_t(k\bar\phi)(0, x)\right)\left(\frac{\partial}{\partial k}V_t(k\bar\phi)(u, x)\right)^2 \right. \\
&\quad \left. -\Phi'(1 - V_t(k\bar\phi)(0, x))\frac{\partial^2}{\partial k^2}V_t(k\bar\phi)(0, x) + \frac{\partial^2}{\partial k^2}V_t(k\bar\phi)(0, x)\right].
\end{aligned}
$$

Letting $\ddot{V}_t\bar\phi = \frac{\partial^2}{\partial k^2}V_t(k\bar\phi)|_{k=0^+}$ we get that

$$
\frac{\partial}{\partial t}\ddot{V}_t\phi(u, x) = \mathcal{L}\ddot{V}_t\bar\phi(u, x) - \lambda(u)\Phi''(1)\left(\dot{V}_t\bar\phi(0, x)\right)^2 - 2\bar\phi(u, x)\dot{V}_t\bar\phi(u, x). \tag{5.9}
$$

From (5.8) and (5.9) we obtain

$$
\ddot{V}_t\bar\phi(u, x) = \int_0^t \tilde{T}_s\left[-\lambda(u)\left(\int_0^s \tilde{T}_r\bar\phi(u, x)dr\right)^2 - 2\bar\phi(u, x)\int_0^s \tilde{T}_r\bar\phi(u, x)dr\right]ds.
$$

Note that $\mathrm{Var}\langle\phi, J_t\rangle = -\langle\ddot{V}_t\bar{\phi}(*, \bullet), F \times \Lambda\rangle$. Therefore,

$$\mathrm{Var}\langle\bar{\phi}, J_t\rangle$$
$$= \left\langle \int_0^t \tilde{T}_s \left[\lambda(*) \left(\int_0^s \tilde{T}_r\bar{\phi}(*, \bullet)dr \right)^2 + 2\bar{\phi}(*, \bullet) \int_0^s \tilde{T}_r\bar{\phi}(*, \bullet) \, dr \right] ds, F \times \Lambda \right\rangle$$
$$= \int_0^t \left\langle \lambda(*) \left(\int_0^s \tilde{T}_r\bar{\phi}(*, \bullet)dr \right)^2, F \times \Lambda \right\rangle ds$$
$$+ 2 \int_0^t \left\langle \bar{\phi}(*, \bullet) \int_0^s \tilde{T}_r\bar{\phi}(*, \bullet)dr, F \times \Lambda \right\rangle ds$$
$$=: \quad (A) + (B). \tag{5.10}$$

Notice that, under the choice of $\bar{\phi}$, $\tilde{T}_t\bar{\phi}(u, x) = \mathcal{S}_t\phi(x)$ for all $t \geq 0$, and that $\langle\lambda, F\rangle < \infty$. In fact, using that $\lambda(u) \sim u^{-1}$ and $f(u) \sim u^{-\gamma-1}$, we get that, for $A > 0$ sufficiently large,

$$\langle\lambda, F\rangle = \int_0^\infty \lambda(u)f(u)du$$
$$= \int_0^A \lambda(u)f(u)du + \int_A^\infty \lambda(u)f(u)du$$
$$\sim \int_0^A \lambda(u)f(u)du + \int_A^\infty u^{-1}u^{-\gamma-1}du$$
$$< \infty.$$

Now,

$$(A) = \int_0^t \left\langle \lambda(*) \left(\int_0^s \mathcal{S}_r\phi(\bullet) \, dr \right)^2, F \times \Lambda \right\rangle ds = \langle\lambda, F\rangle \int_0^t \left\langle \left(\int_0^s \mathcal{S}_r\phi dr \right)^2, \Lambda \right\rangle ds.$$

Also, it can be shown that

$$\int_0^t \left\langle \left(\int_0^s \mathcal{S}_r\phi dr \right)^2, \Lambda \right\rangle ds \leq \mathrm{Const}(\phi_2)(t + t^{3-d/\alpha}),$$

and consequently,

$$(A) \leq \langle\lambda, F\rangle\mathrm{Const}(\phi_2)(t + t^{3-d/\alpha}).$$

Similarly, for the second term in (5.10)

$$(B) = 2 \int_0^t \left\langle \bar{\phi}(*, \bullet) \int_0^s \tilde{T}_r\bar{\phi}(*, \bullet) \, dr, F \times \Lambda \right\rangle ds = 2 \int_0^t \left\langle \int_0^s \mathcal{S}_r\phi dr, \Lambda \right\rangle ds,$$

where

$$\int_0^t \left\langle \int_0^s \mathcal{S}_r\phi \, dr, \Lambda \right\rangle ds \leq \mathrm{Const}(\phi)(t + t^{2-d/\alpha}),$$

hence,

$$(B) \leq 2\mathrm{Const}(\phi)(t + t^{2-d/\alpha}).$$

Finally, combining the bounds for (A) and (B) we obtain the result. $\qquad\square$

6. Proof of Theorem 2.2

Suppose that F is a non-arithmetic distribution function supported on the non-negative real line and having finite mean $\mu > 0$, and let $d > \alpha$. As in the proof of Lemma 5.1, we have that

$$P\left(|T^{-1}\langle\varphi, J_T(1)\rangle - \langle\varphi, \Lambda\rangle| > \epsilon\right) \leq \frac{2}{\epsilon^2} \int_0^1 dv \int_0^v \mathrm{Cov}\left(\langle\varphi, X_{uT}\rangle, \langle\varphi, X_{vT}\rangle\right) du. \tag{6.1}$$

Therefore, due to Proposition 3.3,

$$P\left(|T^{-1}\langle\varphi, J_T(1)\rangle - \langle\varphi, \Lambda\rangle| > \epsilon\right) \leq (I) + (II), \tag{6.2}$$

where

$$(I) := \frac{2}{\epsilon^2} \int_0^1 dv \int_0^v du \langle\varphi \mathcal{S}_{T(v-u)}\varphi, \Lambda\rangle,$$

and

$$(II) := \frac{2}{\epsilon^2} \int_0^1 \int_0^v \int_0^{Tu} \langle(\mathcal{S}_{Tu-r}\varphi)(\mathcal{S}_{Tv-r}\varphi), \Lambda\rangle dU(r) du\, dv.$$

We recall the bound (5.3) for (I). It remains to upper-bound (II). Performing the change of variables $h = r/T$ in (II), and using the elementary renewal theorem (see e.g. Karlin and Taylor, 1975, p. 188), we have that, for T large enough,

$$\begin{aligned}
(II) &= \frac{2}{\epsilon^2} \int_0^1 \int_0^v \int_0^u \langle(\mathcal{S}_{T(u-h)}\varphi)(\mathcal{S}_{T(v-h)}\varphi), \Lambda\rangle d\left[\frac{U(Th)}{Th}Th\right] du\, dv \\
&\sim \frac{2T}{\epsilon^2 \mu} \int_0^1 \int_0^v \int_0^u \langle(\mathcal{S}_{T(u-h)}\varphi)(\mathcal{S}_{T(v-h)}\varphi), \Lambda\rangle dh\, du\, dv \\
&= \frac{2T}{\epsilon^2 \mu} \int_0^1 \int_0^v \int_h^v \langle(\mathcal{S}_{T(u-h)}\varphi)(\mathcal{S}_{T(v-h)}\varphi), \Lambda\rangle du\, dh\, dv.
\end{aligned}$$

After performing several changes of variables one can see that, for all T large enough,

$$\begin{aligned}
(II) &\sim \frac{2}{\epsilon^2 \mu T^2} \int_0^T \int_{\mathbb{R}^d} \int_0^v \int_0^t (\mathcal{S}_s\varphi)(x)(\mathcal{S}_t\varphi)(x) ds\, dt\, dx\, dv \\
&\leq \frac{2}{\epsilon^2 \mu T^2} \int_0^T \int_{\mathbb{R}^d} \int_0^v \int_0^v (\mathcal{S}_s\varphi)(x)(\mathcal{S}_t\varphi)(x) ds\, dt\, dx\, dv \\
&= \frac{2}{\epsilon^2 \mu T^2} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) \int_0^v \int_0^v p_{t+s}(y-z) ds\, dt\, dy\, dz\, dv.
\end{aligned}$$

On the other hand, one can show, as in Méléard and Roelly (1992), that

$$\int_0^v \int_0^v p_{t+s}(y-z) ds\, dt \leq c\left(|y-z|^{2\alpha-d} + v^{2-d/\alpha}\right)$$

for some constant $c > 0$. Hence, for any fixed $A > 0$, and all T large enough,

$$
\begin{aligned}
(II) \quad \leq \quad & \frac{2}{\epsilon^2 \mu T^2} \int_0^A \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) \int_0^v \int_0^v p_{t+s}(y-z) ds\, dt\, dy\, dz\, dv \\
& + \frac{2}{\epsilon^2 \mu T^2} c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z)|y-z|^{2\alpha-d} dy\, dz (T-A) \\
& + \frac{2}{\epsilon^2 \mu T^2} c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) dy\, dz \frac{(T^{3-d/\alpha} - A^{3-d/\alpha})}{3 - d/\alpha} \\
\leq \quad & \frac{2}{\epsilon^2 \mu T^2} \int_0^A \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) \int_0^v \int_0^v p_{t+s}(y-z) ds\, dt\, dy\, dz\, dv \\
& + \frac{2}{\epsilon^2 \mu T} c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z)|y-z|^{2\alpha-d} dy\, dz \\
& + \frac{2}{\epsilon^2 \mu} c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) dy\, dz \frac{(T^{1-d/\alpha} - A^{3-d/\alpha}T^{-2})}{3 - d/\alpha}.
\end{aligned}
$$

The proof concludes with an application of Borel-Cantelli's Lemma, using that $d/\alpha > 1$, (6.2), and the bounds for (I) and (II).

7. Proof of Theorem 2.3

In this section we assume that $d < \alpha\gamma$, and that F is a distribution function satisfying (1.1). Arguing similarly as at the end of the proof of Theorem 2.1, Lemma 7.1 below yields the theorem.

Lemma 7.1. *Let $A \subset \mathbb{R}^d$ be a ball. Then, for all $\epsilon > 0$, and for all t sufficiently large,*

$$
P\left(t^{-1} \int_0^t 1_{\{X_s(A) > 0\}} ds > \epsilon \right) \leq (1 - e^{-\epsilon})^{-1} \left(ct^{-(d/\alpha+\gamma)/2 + (1+\delta)d/\alpha} + c_1 t^{-1} \right),
\tag{7.1}
$$

for some positive constants c and c_1.

Proof: Notice that, by Markov's inequality,

$$
P\left(t^{-1} \int_0^t 1_{\{X_s(A) > 0\}} ds > \epsilon \right) \leq (1 - e^{-\epsilon})^{-1} \mathbb{E}\left[1 - \exp\left\{t^{-1} \int_0^t 1_{\{X_s(A) > 0\}} ds \right\}\right].
\tag{7.2}
$$

Moreover, since the initial population is Poissonian,

$$
\begin{aligned}
\mathbb{E}\left[e^{-t^{-1} \int_0^t 1_{\{X_s(A) > 0\}} ds}\right] \quad = \quad & \mathbb{E}\left[e^{-\int_0^1 1_{\{X_{st}(A) > 0\}} ds}\right] \\
= \quad & \exp\left\{-\int_{\mathbb{R}^d} \mathbb{E}_x\left[1 - e^{-\int_0^1 1_{\{Z_{st}(A) > 0\}} ds}\right] dx\right\}.
\end{aligned}
\tag{7.3}
$$

Now, since $1 - e^{-x} \leq x$, for all $x \geq 0$, we have that

$$
1 - e^{-\int_0^1 1_{\{Z_{st}(A) > 0\}} ds} \leq \int_0^1 1_{\{Z_{st}(A) > 0\}} ds.
$$

Therefore,

$$
\mathbb{E}_x\left[1 - e^{-\int_0^1 1_{\{Z_{st}(A) > 0\}} ds}\right] \leq \mathbb{E}_x \int_0^1 1_{\{Z_{st}(A) > 0\}} ds = \int_0^1 P_x\left(Z_{st}(A) > 0\right) ds.
\tag{7.4}
$$

Due to (7.3) and (7.4), inequality (7.2) can be written as

$$P\left(t^{-1}\int_0^t 1_{\{X_s(A)>0\}}ds > \epsilon\right)$$

$$\leq (1-e^{-\epsilon})^{-1}\left[1-\exp\left\{-\int_0^1\int_{\mathbb{R}^d}P_x\left(Z_{st}(A)>0\right)dx\,ds\right\}\right], \qquad (7.5)$$

where

$$\int_0^1\int_{\mathbb{R}^d}P_x\left(Z_{st}(A)>0\right)dx\,ds = \int_0^1\int_{D(st,\delta)}P_x\left(Z_{st}(A)>0\right)dx\,ds$$

$$+\int_0^1\int_{D(st,\delta)^c}P_x\left(Z_{st}(A)>0\right)dx\,ds \qquad (7.6)$$

with

$$D(t,\delta):=\{x\in\mathbb{R}^d:|x|\leq t^{(1+\delta)/\alpha}\}, \quad \delta>0.$$

Using the inequality

$$\int_{D(st,\delta)}P_x\left(Z_{st}(A)>0\right)dx\leq K(st)^{-(d/\alpha+\gamma)/2+(1+\delta)d/\alpha},$$

which holds for some positive constant K (see Lemma 5 in Vatutin and Wakolbinger, 1999), we deduce that

$$\int_0^1\int_{D(st,\delta)}P_x\left(Z_{st}(A)>0\right)dx\,ds \leq K\int_0^1(st)^{-(d/\alpha+\gamma)/2+(1+\delta)d/\alpha}ds$$

$$= ct^{-(d/\alpha+\gamma)/2+(1+\delta)d/\alpha},$$

where we used that $1-(d/\alpha+\gamma)/2+(1+\delta)d/\alpha>0$. On the other hand, following closely the proof of Lemma 5 in Vatutin and Wakolbinger (1999) one can see that, for sufficiently large t,

$$\int_0^1\int_{\mathbb{R}^d\setminus D(st,\delta)}P_x\left(Z_{st}(A)>0\right)dx\,ds \leq c_1\int_0^1 P\left(\|B_1^0\|\geq\frac{1}{2}(st)^{\delta/\alpha}\right)ds\leq c_1 t^{-1}.$$

In this way, (7.6) yields the inequality

$$\int_0^1\int_{\mathbb{R}^d}P_x\left(Z_{st}(A)>0\right)dx\,ds \leq ct^{-(d/\alpha+\gamma)/2+(1+\delta)d/\alpha}+c_1 t^{-1}, \qquad (7.7)$$

which is valid for all t large enough, and renders (7.1). Notice that $-(d/\alpha+\gamma)/2+(1+\delta)d/\alpha<0$ for sufficiently small δ. $\qquad\square$

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