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On the genealogy of conditioned stable Lévy forests

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Abstract. A Lévy forest of size $s > 0$ is a Poisson point process in the set of Lévy trees which is defined on the time interval $[0, s]$. The total mass of this forest is defined by the sum of the masses of all its trees. We give a realization of the stable Lévy forest of a given size conditioned on its total mass using the path of the unconditioned forest. Then, we prove an invariance principle for this conditioned forest by considering k independent Galton-Watson trees whose offspring distribution is in the domain of attraction of any stable law conditioned on their total progeny to be equal to n. We prove that when n and k tend towards $+\infty$, under suitable rescaling, the associated coding random walk, the contour and height processes all converge in law on the Skorokhod space towards the first passage bridge and height process of a stable Lévy process with no negative jumps respectively.

1. Introduction

The purpose of this work is to study some remarkable properties of stable Lévy forests of a given size conditioned by their mass.

A Galton-Watson tree is the underlying family tree of a given Galton-Watson process with offspring distribution μ started with one ancestor. It is well-known that if μ is critical or subcritical, the Galton-Watson process reaches 0 in a finite time a.s. and therefore, the corresponding Galton-Watson tree is a.s. finite. In this case, Galton-Watson trees can be coded by two different discrete real valued processes: the height process and the contour process. The definitions of the latter two processes are in Section 2. Both processes describe the genealogical structure of the

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associated Galton-Watson process. They are not Markovian in general (however the contour is a reflected random walk when the offspring distribution is geometric). In any case, these processes can be written as functionals of a certain left-continuous random walk, also coding the genealogy of the tree, whose jump distribution depends on the offspring distribution μ. In a natural way, Galton-Watson forests are finite or infinite collections of independent Galton-Watson trees. The size of a finite Galton-Watson forest is just the numbers of trees that it contains and its mass is the total number of its vertices.

Our presentation of Lévy trees owes a lot to the recent paper by Duquesne and Le Gall (2005), which uses the formalism of $\mathbb{R}$-trees that was implicit in Duquesne (2003), Duquesne and Le Gall (2002) and Le Gall and Le Jan (1998). We may consider Lévy trees as random variables taking values in the space of compact rooted $\mathbb{R}$-trees. Their definition bears upon the continuous analogue of the height process of Galton-Watson trees introduced by Le Gall and Le Jan (1998) as a functional of a Lévy process with no negative jumps. In a recent paper of Evans et al. (2006), $\mathbb{R}$-trees are studied from the point of view of measure theory. Informally an $\mathbb{R}$-tree is a metric space $(\mathcal{T}, d)$ such that for any two points σ and σ' in $\mathcal{T}$ there is a unique arc with endpoints σ and σ' and furthermore this arc is isometric to a compact interval of the real line. In Evans et al. (2006), the authors also established that the space $\mathbb{T}$ of equivalent classes of (rooted) compact real trees, endowed with the Gromov-Hausdorff metric, is a Polish space. This makes it very natural to consider random variables or even random processes taking values in the space $\mathbb{T}$. In this work, we define Lévy forests as Poisson point processes with values in the set of $\mathbb{R}$-trees whose intensity measure is the law of the generic Lévy tree. A Lévy forest with finite size $s > 0$ is then such a Poisson point process on the time interval $[0, s]$.

The local time at level a of a Lévy tree $(\mathcal{T}, d)$ is a finite measure ℓ^a supported on the level set $\mathcal{T}(a) = \{v \in \mathcal{T} : d(\rho(\mathcal{T}), v) = a\}$, where $\rho(\mathcal{T})$ is the root of $\mathcal{T}$. The total mass of $\mathcal{T}$ is then defined by $\int_0^\infty da\ell^a$ and the total mass of a Lévy forest with finite size is the sum of the masses of all its trees. First, we are interested in the construction of Lévy forests of a given size conditioned by their mass. Again, in the discrete setting this conditioning is easier to define; the conditioned Galton-Watson forest of size k and mass n is a collection of k independent Galton-Watson trees with total progeny equal to n. In section 4, we provide a definition of these notions for Lévy forest. Then, in the stable case, we give a construction of the conditioned stable Lévy forest of size $s > 0$ and mass 1 by "rescaling" the unconditioned forest of a particular random mass.

In Aldous (1991), the author showed that the Brownian random tree (or continuum random tree) is the limit as n increases of a rescaled critical Galton-Watson tree conditioned to have n vertices whose offspring distribution has a finite variance. In particular, Aldous proved that the discrete height process converges on the Skorokhod space of càdlàg paths to the normalized Brownian excursion. Recently, Duquesne (2003) extended such results to Galton-Watson trees whose offspring distribution is in the domain of attraction of a stable law with index α in $(1, 2]$. Then, Duquesne showed that the discrete height process of the Galton-Watson tree conditioned to have a deterministic progeny, converges as this progeny tends to infinity on the Skorokhod space to the normalized excursion of the height process associated with the stable Lévy process.

The other main purpose of our work is to study this convergence in the case of a finite number of independent Galton-Watson trees, this number being an increasing function of the progeny. More specifically, in Section 5, we establish an invariance principle for the conditioned forest by considering k independent Galton-Watson trees whose offspring distribution is in the domain of attraction of any stable law conditioned on their total progeny to be equal to n. When n and k tend towards ∞, under suitable rescaling, the associated coding random walk, the contour and height processes converge in law on the space of Skorokhod towards the first passage bridge of a stable Lévy process with no negative jumps and its height process.

In section 2, we introduce conditioned Galton-Watson forests and their related coding by a first passage bridge of the associated random walk, by the height process and by the contour process. Section 3 is devoted to recalling the definitions of real trees and Lévy trees and stating a number of important results related to these notions.

2. Discrete trees and forests.

In this section, we first recall Ulam's coding of rooted ordered trees, see Neveu (1986). Then we state some preliminary results in discrete time.

In all the sequel, an element u of $(\mathbb{N}^*)^n$ is written as $u = (u_1, \dots u_n)$ and we set $|u| = n$. Let

$$\mathbb{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where $\mathbb{N}^* = \{1, 2, \dots\}$ and by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. The concatenation of two elements of $\mathbb{U}$, let us say $u = (u_1, \dots u_n)$ and $v = (v_1, \dots, v_m)$ is denoted by $uv = (u_1, \dots u_n, v_1, \dots, v_m)$. A discrete rooted tree is an element τ of the set $\mathbb{U}$ which satisfies:

(i) $\emptyset \in \tau$,

(ii) If $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in \tau$,

(iii) For every $u \in \tau$, there exists a number $k_u(\tau) \geq 0$, such that $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

In this definition, $k_u(\tau)$ represents the number of children of the vertex u. We denote by $\mathbb{T}$ the set of all rooted trees. The total cardinality of an element $\tau \in \mathbb{T}$ will be denoted by $\zeta(\tau)$, (we emphasize that the root is counted in $\zeta(\tau)$). If $\tau \in \mathbb{T}$ and $u \in \tau$, then we define the shifted tree at the vertex u by

$$\theta_u(\tau) = \{v \in \mathbb{U} : uv \in \tau\}\,.$$

A vertex w is an ancestor of u if there exists $x \in \mathbb{U}$ such that $u = wx$. We denote by $u \wedge v$ the last common ancestor of the vertices u and v according to the lexicographical order.

Then we consider a probability measure μ on $\mathbb{N}$, such that

$$\sum_{k=0}^{\infty} k\mu(k) \leq 1 \quad \text{and} \quad \mu(0) + \mu(1) < 1\,.$$

Let us endow $\mathbb{T}$ with the sigma-field $\mathcal{P}(\mathbb{T})$ of all its subsets. The law of the Galton-Watson tree with offspring distribution μ is the unique probability measure $\mathbb{Q}_\mu$ on $(\mathbb{T}, \mathcal{P}(\mathbb{T}))$ such that:

(i) $\mathbb{Q}_\mu(k_\emptyset = j) = \mu(j)$, $j \in \mathbb{Z}_+$.

(*ii*) For every $j \geq 1$, with $\mu(j) > 0$, the shifted trees $\theta_1(\tau), \dots, \theta_j(\tau)$ are independent under the conditional distribution $\mathbb{Q}_\mu(\,\cdot\,|\,k_\emptyset = j)$ and their conditional law is $\mathbb{Q}_\mu$.

The Galton-Watson process associated to a Galton-Watson tree is the Markov chain (Z_n) indexed by the generations, such that Z_n is the number of individuals in the tree at generation n. In Neveu (1986), a construction of the law of (Z_n) is given in terms of the measure $\mathbb{Q}_\mu$. Clearly the Galton-Watson process does not code entirely the genealogy of the tree. In the aim of doing so, other (coding) real valued processes have been defined. Amongst such processes one can cite the *height process*, the *contour process* and the associated random walk which will be called here *the coding walk* and which is sometimes referred to as the Luckazievicks path.

In order to define these processes, let us first denote by $u_\tau(0) = \emptyset$, $u_\tau(1) = 1, \dots, u_\tau(\zeta-1)$ the elements of a tree τ which are enumerated in the lexicographical order (when no confusion is possible, we will simply write $u(n)$ for $u_\tau(n)$). Let us also denote by $|u(n)|$ the rank of the generation of a vertex $u(n) \in \tau$. The following definitions may be found in Le Gall and Le Jan (1998).

Definition 2.1. The height function of a tree τ is:

$$H_n(\tau) = |u(n)|,\; 0 \leq n \leq \zeta(\tau) - 1\,.$$

Suppose that the tree τ is embedded in the half-plane in such a way that edges have length one. Then starting at time $t = 0$ from the root we run along the tree from the left to the right, moving continuously along the edges at unit speed, until we come back to the root. For $t \in [0, 2(\zeta(\tau) - 1))]$ the value $C_t(\tau)$ of the contour function is the distance (on the tree) between our position at time t and the root. Then we set $C_t(\tau) = 0$, for $t \in [2(\zeta(\tau) - 1), 2\zeta(\tau)]$.

The coding walk $S(\tau)$ is the discrete time process whose increments are:

$$S_0 = 0\,, \quad S_{n+1}(\tau) - S_n(\tau) = k_{u(n)}(\tau) - 1, \quad 0 \leq n \leq \zeta(\tau) - 1\,.$$

A Galton-Watson forest with offspring distribution μ is a finite or infinite sequence of independent Galton-Watson trees with offspring distribution μ. It will be denoted by $\mathcal{F} = (\tau_k)$. With a misuse of notation, we will denote by $\mathbb{Q}_\mu$ the law on $(\mathbb{T})^{\mathbb{N}^*}$ of a Galton-Watson forest with offspring distribution μ.

The height function of a forest $\mathcal{F} = (\tau_k)$ is obtained from the concatenation of the height functions $H(\tau_1), H(\tau_2), \dots, H(\tau_k), \dots$. The same construction holds for the contour function and the coding walk of $\mathcal{F}$ that are respectively denoted $H(\mathcal{F})$, $C(\mathcal{F})$ and $S(\mathcal{F})$. Let us give a more formal definition.

Definition 2.2. With the convention that $\zeta(\tau_0) = 0$, the processes $H(\mathcal{F})$, $C(\mathcal{F})$ and $S(\mathcal{F})$ are defined as follows:

$$\begin{aligned} H_n(\mathcal{F}) &= H_{n-(\zeta(\tau_0)+\dots+\zeta(\tau_{k-1}))}(\tau_k), \\ &\quad \text{if } \; \zeta(\tau_0) + \dots + \zeta(\tau_{k-1}) \leq n \leq \zeta(\tau_0) + \dots + \zeta(\tau_k) - 1, \text{ for } k \geq 1. \end{aligned}$$

If there are $j < \infty$ trees in the forest, then we set $H_n(\mathcal{F}) = 0$, for $n \geq \zeta(\tau_0) + \dots + \zeta(\tau_j)$.

$$\begin{aligned} C_t(\mathcal{F}) &= C_{t-2(\zeta(\tau_0)+\dots+\zeta(\tau_{k-1}))}(\tau_k), \\ &\quad \text{if } 2(\zeta(\tau_0) + \dots + \zeta(\tau_{k-1})) \leq t \leq 2(\zeta(\tau_0) + \dots + \zeta(\tau_k)), \text{ for } k \geq 1. \end{aligned}$$

If there are $j < \infty$ trees in the forest, then we set $C_t(\mathcal{F}) = 0$, for $t \geq 2(\zeta(\tau_0) + \cdots + \zeta(\tau_j))$.

$$S_n(\mathcal{F}) = S_{n-(\zeta(\tau_0)+\cdots+\zeta(\tau_{k-1}))}(\tau_k) - k + 1,$$
$$\text{if } \ \zeta(\tau_0) + \cdots + \zeta(\tau_{k-1}) \leq n \leq \zeta(\tau_0) + \cdots + \zeta(\tau_k), \text{ for } k \geq 1.$$

If there are $j < \infty$ trees in the forest, then we set $S_n(\mathcal{F}) = S_{\zeta(\tau_0)+\cdots+\zeta(\tau_j)}(\mathcal{F})$, for $n \geq \zeta(\tau_0) + \cdots + \zeta(\tau_j)$.

It is not difficult to check that each of these three processes allows us to recover the entire structure of the forest. We say that they *codes* the genealogy of the forest. Moreover, it is well known that when the number of trees is infinite, the coding walk $S(\mathcal{F})$ is a downward skip free random walk with initial value $S_0 = 0$ and step distribution $\nu(k) = \mu(k+1)$, $k = -1, 0, 1, \ldots$. We may easily see on a picture that the k-th excursion above its past minimum of $S(\mathcal{F})$ corresponds to $S(\tau_k)$.
Let us denote $H(\mathcal{F})$, $C(\mathcal{F})$ and $S(\mathcal{F})$ respectively by H, C and S when no confusion is possible. In the sequel, we will have to use some path relationships between H, C and S that we recall now. Let us suppose that $\mathcal{F}$ is infinite. It is established for instance in Duquesne and Le Gall (2002); Le Gall and Le Jan (1998) that

$$H_n = \text{card}\left\{0 \leq k \leq n-1 : S_k = \inf_{k \leq j \leq n} S_j\right\}. \tag{2.1}$$

This identity means that the height process at each time n can be interpreted as the amount of time that the random walk S spends at its future minimum before n. The following relationship between H and C is proved in Duquesne and Le Gall (2002), set $K_n = 2n - H_n$:

$$C_t = \begin{cases} (H_n - (t - K_n))^+, & \text{if } t \in [K_n, K_{n+1} - 1] \\ (H_{n+1} - (K_{n+1} - t))^+, & \text{if } [K_{n+1} - 1, K_{n+1}]. \end{cases} \tag{2.2}$$

As preliminary results, we also state two inequalities that are proved in Duquesne and Le Gall (2002), section 2.4 and can easily be deduced from (2.2). Define the function $f : \mathbb{R}_+ \to \mathbb{Z}_+$ by $f(t) = n$ if and only if $t \in [K_n, K_{n+1})$. For every integer $n \geq 1$,

$$\sup_{t \in [0, K_n]} |C_t - H_{f(t)}| \leq 1 + \sup_{k \leq n} |H_{k+1} - H_k|,$$
$$\sup_{t \in [0, K_n]} |f(t) - \frac{t}{2}| \leq \frac{1}{2} \sup_{k \leq n} H_k + 1. \tag{2.3}$$

The starting point of our work is the observation that a Galton-Watson forest with k trees conditioned to have n vertices can be coded by a downward skip free random walk conditioned to first reach $-k$ at time n. An interpretation of this result may be found in Pitman (2006), Lemma 6.3 for instance.

Proposition 2.3. *Let $\mathcal{F} = (\tau_j)$ be an infinite forest with offspring distribution μ and S be its coding walk. Let W be a random walk defined on a probability space $(\Omega, \mathcal{F}, P)$ with the same law as S. Define $T_i^W = \inf\{j : W_j = -i\}$, for $i \geq 1$ and choose k and n such that $P(T_k^W = n) > 0$.*

Under the conditional law $\mathbb{Q}_\mu(\cdot \,|\, \zeta(\tau_1) + \cdots + \zeta(\tau_k) = n)$, the process $(S_j,\, 0 \leq j \leq \zeta(\tau_1) + \cdots + \zeta(\tau_k))$ has the same law as the process $(W_j,\, 0 \leq j \leq T_k^W)$ under $P(\cdot \,|\, T_k^W = n)$.

In the proof of Theorem 5.1 (see section 5) condition $P(T_k^W = n) > 0$ must be satisfied for all k and n sufficiently large such that $1 \leq k \leq n$. We remark that this is the case if W is aperiodic, since from Kemperman's formula (see Pitman, 2006, page 122) we have for any downward skip free random walk:

$$P(T_k^W = n) = \frac{k}{n} P(W_n = -k)\,. \tag{2.4}$$

To end this section, note that the identity in law of the above proposition also holds clearly for the triple (S, H, C), with obvious definitions for H^W and C^W. In Section 4, we will present a continuous time version of this result, but before we need to introduce the continuous time setting of Lévy trees and forests.

3. Coding real trees and forests

Discrete trees may be considered in an obvious way as compact metric spaces with no loops. Such metric spaces are special cases of $\mathbb{R}$-trees that are defined hereafter. Similarly to the discrete case, an $\mathbb{R}$-forest is any countable collection of $\mathbb{R}$-trees. In this section we keep the same notation as in Duquesne and Le Gall (2002) and Duquesne and Le Gall (2005). The following formal definition of $\mathbb{R}$-trees is now standard and originates from T-theory. It may be found for instance in Dress et al. (1996) or Evans (2008).

Definition 3.1. A metric space $(\mathcal{T}, d)$ is an $\mathbb{R}$-tree if for every $\sigma_1, \sigma_2 \in \mathcal{T}$,

1. There is a unique map f_{σ_1,σ_2} from $[0, d(\sigma_1, \sigma_2)]$ into $\mathcal{T}$ such that $f_{\sigma_1,\sigma_2}(0) = \sigma_1$ and $f_{\sigma_1,\sigma_2}(d(\sigma_1, \sigma_2)) = \sigma_2$.
2. If g is a continuous injective map from $[0, 1]$ into $\mathcal{T}$ such that $g(0) = \sigma_1$ and $g(1) = \sigma_2$, we have
$$g([0, 1]) = f_{\sigma_1,\sigma_2}([0, d(\sigma_1, \sigma_2)])\,.$$

A rooted $\mathbb{R}$-tree is an $\mathbb{R}$-tree $(\mathcal{T}, d)$ with a distinguished vertex $\rho = \rho(\mathcal{T})$ called the root. An $\mathbb{R}$-forest is any countable collection of rooted $\mathbb{R}$-trees: $\mathcal{F} = \{(\mathcal{T}_i, d_i),\, i \in \mathcal{I}\}$.

A construction of some particular cases of such metric spaces has been given by Aldous (1991) and is described in Duquesne and Le Gall (2005) in a more general setting. For $a > 0$, let $f : [0, a] \to [0, \infty)$ be a continuous function such that $f(0) = f(a) = 0$. For $0 \leq s \leq t \leq a$, we define

$$d_f(s, t) = f(s) + f(t) - 2 \inf_{u \in [s,t]} f(u) \tag{3.1}$$

and the equivalence relation by

$$s \sim t \quad \text{if and only if} \quad d_f(s, t) = 0\,.$$

(Note that $d_f(s, t) = 0$ if and only if $f(s) = f(t) = \inf_{u \in [s,t]} f(u)$.) Then the projection of d_f on the quotient space

$$\mathcal{T}_f = [0, a]/\sim$$

defines a distance that will also be denoted by d_f. The metric space $(\mathcal{T}_f, d_f)$ is then a compact $\mathbb{R}$-tree, see for instance Duquesne and Le Gall (2005) and Evans (2008). Denote by $p_f : [0, a] \to \mathcal{T}_f$ the canonical projection. The vertex $\rho = p_f(0)$ will be chosen as the root of $\mathcal{T}_f$.

The space of $\mathbb{R}$-trees will be denoted by $\mathbb{T}_c$. It is endowed with the Gromov-Hausdorff distance, d_{GH} that we briefly recall now. For a metric space (E,δ) and K, K' two compact subsets of E, $\delta_{\text{Haus}}(K,K')$ will denote the Hausdorff distance between K and K', i.e. $\delta_{\text{Haus}}(K,K')=\inf\{r>0 : A\subset U_r(B) \text{ and } B\subset U_r(A)\}$, where for a subset S of E, $U_r(S)=\{x\in E:\delta(x,S)<r\}$ and $\delta(x,S)=\inf\{\delta(x,y): y\in S\}$. Then we define the distance between two compact rooted $\mathbb{R}$-trees $\mathcal{T}$ and $\mathcal{T}'$ by:

$$d_{GH}(\mathcal{T},\mathcal{T}')=\inf\left(\delta_{\text{Haus}}(\varphi(\mathcal{T}),\varphi'(\mathcal{T}'))\vee\delta(\varphi(\rho),\varphi'(\rho'))\right),$$

where the infimum is taken over all isometric embeddings $\varphi:\mathcal{T}\to E$ and $\varphi':\mathcal{T}'\to E$ of $\mathcal{T}$ and $\mathcal{T}'$ into a common metric space (E,δ). We refer to Gromov (1999) for a complete description of the Gromov-Hausdorff topology, see also Evans et al. (2006) and Chapter 4 of Evans (2008). It is important to note that the space $(\mathbb{T}_c,d_{GH})$ is complete and separable, see for instance Theorem 3.23 of Evans (2008) or Theorem 1 of Evans et al. (2006).

In the remainder of this section, we will recall from Duquesne and Le Gall (2005) the definition of Lévy trees and of Lévy forests. Let $(\mathbb{P}_x)$, $x\in\mathbb{R}$ be a family of probability measures on the Skorokhod space $\mathbb{D}$ of càdlàg paths from $[0,\infty)$ to $\mathbb{R}$ such that for each $x\in\mathbb{R}$, the canonical process X is a Lévy process with no negative jumps, starting from x, that does not drift to $+\infty$. (Set $\mathbb{P}=\mathbb{P}_0$, so $\mathbb{P}_x$ is the law of $X+x$ under $\mathbb{P}$.) This is equivalent to assume that the characteristic exponent ψ of X (i.e. $\mathbb{E}(e^{-\lambda X_t})=e^{t\psi(\lambda)}$, $\lambda\in\mathbb{R}_+$) is of the form

$$\psi(\lambda)=\alpha\lambda+\beta\lambda^2+\int_{(0,\infty)}(e^{-\lambda r}-1+\lambda r)\,\pi(dr)\,,$$

(where $\alpha,\beta\geq 0$ and π is a σ-finite measure on $(0,\infty)$ such that $\int(1\wedge r^2)\,\pi(dr)<\infty$) and satisfies the following condition:

$$\int^{\infty}\frac{du}{\psi(u)}<\infty\,. \tag{3.2}$$

This condition is equivalent to the a.s. extinction of the continuous state branching process that is associated to the branching mechanism ψ. Equivalently, the Lévy tree that is constructed from the process X (see below) is a.s. finite. In the sequel we will only consider the stable case where $\psi(\lambda)=\lambda^\alpha$, with $\alpha\in(1,2]$. Condition (3.2) is then obviously satisfied.

By analogy with the discrete case, the continuous time height process $\bar{H}$ is the measure (in a sense which is to be defined) of the set $\{s\leq t: X_s=\inf_{s\leq r\leq t}X_r\}$. A rigorous meaning to this measure is given by the following result due to Le Gall and Le Jan (1998), see also Duquesne and Le Gall (2002). Define $I_t^s=\inf_{s\leq u\leq t}X_u$. There is a sequence of positive real numbers (ε_k) which decreases to 0 such that for any t, the limit

$$\bar{H}_t\overset{(\mathrm{def})}{=}\lim_{k\to+\infty}\frac{1}{\varepsilon_k}\int_0^t\mathbf{1}_{\{X_s-I_t^s<\varepsilon_k\}}\,ds \tag{3.3}$$

exists a.s. It is also proved in Le Gall and Le Jan (1998) that under assumption (3.2), $\bar{H}$ is a continuous process, so that each of its positive excursions codes a real tree in the sense of Aldous. We easily deduce from this definition that the height process $\bar{H}$ is a functional of the Lévy process reflected at its minimum, i.e. $X-I$, where $I:=I^0$. The process $X-I$ is strongly Markovian and under our assumptions, 0 is regular for itself for this process, moreover $-I$ is a local time at level 0. Let us notice that when X is a scaled Brownian motion, i.e. $\psi(\lambda)=\beta\lambda^2$, the process $\bar{H}$ is almost surely equal to the scaled reflected process at its maximum: $\beta^{-1}(M-X)$, where $M_t = \sup_{s\le t} X_s$.

In order to define the Lévy forest, we need to introduce the local times of the height process $\bar{H}$. It is proved in Duquesne and Le Gall (2002) that for any level $a \ge 0$, there exists a continuous increasing process $(L_t^a,\, t\ge 0)$ which is defined by the approximation:

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left(\sup_{0\le s\le t}\left|\frac{1}{\varepsilon}\int_0^s du\mathbf{1}_{\{a<\bar{H}_u\le a+\varepsilon\}} - L_s^a\right|\right) = 0\,. \tag{3.4}$$

The support of the measure dL_t^a is contained in the set $\{t\ge 0 : \bar{H}_t = a\}$ and, as we readily noticed, $L^0 = -I$. Then we may define the Poisson point process of the excursions away from 0 of the process $\bar{H}$ as follows. Let $T_u = \inf\{t : -I_t \ge u\}$ be the right continuous inverse of the local time at 0 of the reflected process $X - I$ (or equivalently of H). The time T_u corresponds to the first passage time of X bellow $-u$. Set $T_{0-} = 0$ and for all $u \ge 0$,

$$e_u(v) = \begin{cases} \bar{H}_{T_{u-}+v}\,, & \text{if } 0\le v\le T_u - T_{u-} \\ 0\,, & \text{if } v > T_u - T_{u-} \end{cases}\,.$$

For each $u\ge 0$, we define the random $\mathbb{R}$-tree $(\mathcal{T}_{e_u}, d_{e_u})$ under $\mathbb{P}$ as in the beginning of this section. We easily deduce from the Markov property of $X - I$ that under the probability measure $\mathbb{P}$, the process $\{(\mathcal{T}_{e_u}, d_{e_u}), u\ge 0\}$ is a Poisson point process in $\mathbb{T}_c$. Let us denote by $\Theta(d\mathcal{T})$ its (σ-finite) intensity measure on $\mathbb{T}_c$.

Definition 3.2. The Lévy forest $\mathcal{F}_{\bar{H}}$ is the Poisson point process

$$(\mathcal{F}_{\bar{H}}(u),\, u\ge 0) \overset{\text{(def)}}{=} \{(\mathcal{T}_{e_u}, d_{e_u}), u\ge 0\}$$

whose intensity measure under $\mathbb{P}$ is $\Theta(d\mathcal{T})$. For each $s>0$, the process

$$\mathcal{F}_{\bar{H}}^s \overset{\text{(def)}}{=} \{(\mathcal{T}_{e_u}, d_{e_u}), 0\le u\le s\}$$

under $\mathbb{P}$ will be called the Lévy forest of *size s*. The Lévy tree $(\mathcal{T}_{\bar{H}}, d_{\bar{H}})$ is a generic point of $\mathcal{F}_{\bar{H}}$, i.e. a $\mathbb{T}_c$-valued random variable with law $\Theta(d\mathcal{T})$.

Such a definition of a Lévy forest has already been introduced in Pitman (2006), Proposition 7.8 in the Brownian setting. However in this work, it is observed that the Brownian forest may also simply be defined as the real tree coded by the function $\bar{H}$ under law $\mathbb{P}$. We also refer to Pitman and Winkel (2005) where the Brownian forest is understood in this way. Similarly, the Lévy forest with size s may be defined as the compact real tree coded by the continuous function with compact support $(\bar{H}_u,\, 0\le u\le T_s)$ under law $\mathbb{P}$. These definitions are more natural when considering convergence of sequences of real forests and we will make appeal to them in section 5, see Corollary 5.3.

We will simply denote the Lévy tree and the Lévy forest respectively by $\mathcal{T}_{\bar{H}}$, $\mathcal{F}_{\bar{H}}$ or $\mathcal{F}_{\bar{H}}^s$, the corresponding distances being implicit. When X is stable, condition (3.2) is satisfied if and only if its index α satisfies $\alpha\in(1,2]$. We may check, as a consequence of (3.3), that $\bar{H}$ is a self-similar process with index $\alpha/(\alpha-1)$, i.e.:

$$(\bar{H}_t,\, t\ge 0) \overset{(d)}{=} (k^{(\alpha-1)/\alpha}\bar{H}_{kt},\, t\ge 0)\,, \quad \text{for all } k>0.$$

(See Duquesne and Le Gall, 2002, section 3.3). In this case, the Lévy tree $\mathcal{T}_{\bar{H}}$ associated to the stable mechanism is called the α-stable Lévy tree and its law is denoted by $\Theta_\alpha(d\mathcal{T})$. This random metric space also inherits from X a scaling property which may be stated as follows: for any $a > 0$, we denote by $a\mathcal{T}_{\bar{H}}$ the Lévy tree $\mathcal{T}_{\bar{H}}$ endowed with the distance $ad_{\bar{H}}$, i.e.

$$a\mathcal{T}_{\bar{H}} \overset{\text{(def)}}{=} (\mathcal{T}_{\bar{H}}, ad_{\bar{H}})\,. \tag{3.5}$$

Then the law of $a\mathcal{T}_{\bar{H}}$ under $\Theta_\alpha(d\mathcal{T})$ is $a^{\frac{1}{\alpha-1}}\Theta_\alpha(d\mathcal{T})$. This property is stated in Le Gall (2006) Proposition 4.3 and Duquesne and Le Gall (2006) where other fractal properties of stable trees are considered.

4. Construction of the conditioned Lévy forest

In this section we present the continuous analogue of discrete forests introduced in section 2. In particular, we define the total mass of the Lévy forest of a given size s. Then we define the Lévy forest of size s conditioned by its total mass. In the stable case, we give a construction of this conditioned forest from the unconditioned forest.

We begin with the definition of the measure $\ell^{a,u}$ which represents a local time at level $a > 0$ for the Lévy tree $\mathcal{T}_{e_u}$. For all $a > 0$, $u \geq 0$ and for every bounded and continuous function φ on $\mathcal{T}_{e_u}$, the finite measure $\ell^{a,u}$ is defined by:

$$\langle \ell^{a,u}, \varphi \rangle = \int_0^{T_u - T_{u-}} dL^a_{T_{u-}+v}\varphi(p_{e_u}(v))\,, \tag{4.1}$$

where we recall from the previous section that p_{e_u} is the canonical projection from $[0, T_u - T_{u-}]$ onto $\mathcal{T}_{e_u}$ for the equivalence relation $\sim$ and (L^a_u) is the local time at level a of $\bar{H}$. Then the mass measure $\mathbf{m}_u$ is defined as the image of the Lebesgue measure on $[0, T_u - T_{u-}]$ under the mapping $v \mapsto p_{e_u}(v)$. There exists a càdlàg version of the mapping $a \mapsto \ell^{a,u}$, see Theorem 4.3 in Duquesne and Le Gall (2005), so that from (4.1) the total mass of the Lévy tree $\mathcal{T}_{e_u}$ can be expressed as:

$$\mathbf{m}_u(\mathcal{T}_{e_u}) = \int_0^\infty da\, \ell^{a,u}. \tag{4.2}$$

Now we fix $s > 0$; the total mass of the forest of size s, $\mathcal{F}^s_{\bar{H}}$ is naturally given by

$$\mathbf{M}_s = \sum_{0 \leq u \leq s} \mathbf{m}_u(\mathcal{T}_{e_u})\,.$$

The total mass $\mathbf{m}_u(\mathcal{T}_{e_u})$ of each tree $\mathcal{T}_{e_u}$ being $T_u - T_{u-}$, this implies that

$$T_s = \mathbf{M}_s\,, \quad \mathbb{P}\text{-a.s.} \tag{4.3}$$

Then we will construct processes which encode the genealogy of the Lévy forest of size s conditioned to have a mass equal to $t > 0$. From the analogy with the discrete case in Proposition 2.3, the natural candidates may informally be defined as:

$$\begin{aligned} X^{br} &\overset{\text{(def)}}{=} [(X_u,\, 0 \leq u \leq T_s)\,|\, T_s = t] \\ \bar{H}^{br} &\overset{\text{(def)}}{=} [(\bar{H}_u,\, 0 \leq u \leq T_s)\,|\, T_s = t]\,. \end{aligned}$$

When X is the Brownian motion, the process X^{br} is called the *first passage bridge*, see Bertoin et al. (2003). In order to give a proper definition in the general case, we need the additional assumption:

The semigroup of $(X, \mathbb{P})$ is absolutely continuous with respect to the Lebesgue measure.

Then denote by $p_t(\cdot)$ the density of the semigroup of X, by $\mathcal{G}_u^X \stackrel{\text{(def)}}{=} \sigma\{X_v, v \le u\}$, $u \ge 0$ the σ-field generated by X and set $\hat{p}_t(x) = p_t(-x)$.

Lemma 4.1. *The probability measure which is defined on each* $\mathcal{G}_u^X$*,* $u \in [0,t)$ *by*

$$\mathbb{P}(X^{br} \in \Lambda_u) = \mathbb{E}\left(\mathbf{1}_{\{X\in\Lambda_u,\, u<T_s\}} \frac{t(s+X_u)}{s(t-u)} \frac{\hat{p}_{t-u}(s+X_u)}{\hat{p}_t(s)}\right), \quad u<t, \quad \Lambda_u \in \mathcal{G}_u^X, \tag{4.4}$$

is a regular version of the conditional law of $(X_u,\, 0 \le u \le T_s)$ *given* $T_s = t$*, in the sense that for all* $u > 0$*, for* λ*-a.e.* $s > 0$ *and* λ*-a.e.* $t > u$*,*

$$\mathbb{P}(X^{br} \in \Lambda_u) = \lim_{\varepsilon\downarrow 0} \mathbb{P}(X \in \Lambda_u \,|\, |T_s - t| < \varepsilon),$$

where λ *is the Lebesgue measure.*

Proof: Let $u < t$, $\Lambda_u \in \mathcal{G}_u^X$ and $\varepsilon < t - u$. From the Markov property, we may write

$$\begin{aligned}\mathbb{P}(X \in \Lambda_u \,|\, |T_s - t| < \varepsilon) &= \mathbb{E}\left(\mathbf{1}_{\{X\in\Lambda_u\}} \frac{\mathbf{1}_{\{\,|T_s-t|<\varepsilon\}}}{\mathbb{P}(|T_s-t|<\varepsilon)}\right)\\ &= \mathbb{E}\left(\mathbf{1}_{\{X\in\Lambda_u, u<T_s\}} \frac{\mathbb{P}_{X_u}(|T_s-(t-u)|<\varepsilon)}{\mathbb{P}(|T_s-t|<\varepsilon)}\right).\end{aligned} \tag{4.5}$$

On the other hand, from Corollary VII.3 in Bertoin (1996) one has for λ-a.e. $s > 0$,

$$t\mathbb{P}(T_s \in dt) = s\hat{p}_t(s)\,dt. \tag{4.6}$$

Hence, for all $x \in \mathbb{R}$, for all $u > 0$, for λ-a.e. $s > 0$ and λ-a.e. $t > u$,

$$\lim_{\varepsilon\downarrow 0} \frac{\mathbb{P}_x(|T_s-(t-u)|<\varepsilon)}{\mathbb{P}(|T_s-t|<\varepsilon)} = \frac{t(s+x)}{s(t-u)} \frac{\hat{p}_{t-u}(s+x)}{\hat{p}_t(s)}. \tag{4.7}$$

Moreover from (4.6) and the Markov property we have

$$\mathbb{E}\left(\mathbf{1}_{\{u<T_s\}} \frac{t(s+X_u)}{s(t-u)} \frac{\hat{p}_{t-u}(s+X_u)}{\hat{p}_t(s)}\right) = 1, \tag{4.8}$$

for λ-a.e. t. Then (4.5), (4.7) and Fatou's lemma imply that

$$\liminf_{\varepsilon\downarrow 0} \mathbb{P}(X \in \Lambda_u \,|\, |T_s-t|<\varepsilon) \ge \mathbb{E}\left(\mathbf{1}_{\{X\in\Lambda_u,\, u<T_s\}} \frac{t(s+X_u)}{s(t-u)} \frac{\hat{p}_{t-u}(s+X_u)}{\hat{p}_t(s)}\right).$$

But replacing Λ_u by Λ_u^c in the above inequality and using (4.8) gives

$$\limsup_{\varepsilon\downarrow 0} \mathbb{P}(X \in \Lambda_u \,|\, |T_s-t|<\varepsilon) \le \mathbb{E}\left(\mathbf{1}_{\{X\in\Lambda_u,\, u<T_s\}} \frac{t(s+X_u)}{s(t-u)} \frac{\hat{p}_{t-u}(s+X_u)}{\hat{p}_t(s)}\right),$$

which ends the proof of the lemma. □

From Lemma 4.1 and a monotone class argument, it follows that the law of X^{br} admits a unique extension on the σ-field $\mathcal{G}_{t-}^X$ and $\mathbb{P}(X_{t-}^{br} = -s) = 1$. We define the law of X^{br} on $\mathcal{G}_t^X$, by setting $X_t^{br} = -s$. Then note that H is a $\mathcal{G}^X$-adapted process, so we can use Lemma 4.1 and (3.3) to construct the law of a height process $\bar{H}^{br}$ associated to the first passage bridge X^{br}. More specifically, the law of $\bar{H}^{br}$ on each $\mathcal{G}_u^X$, $u \in [0,t)$ is given by

$$\mathbb{P}(\bar{H}^{br} \in \Lambda_u) = \mathbb{E}\left(\mathbf{1}_{\{\bar{H}\in\Lambda_u,\, u<T_s\}} \frac{t(s+X_u)}{s(t-u)} \frac{\hat{p}_{t-u}(s+X_u)}{\hat{p}_t(s)}\right), \quad u<t, \quad \Lambda_u \in \mathcal{G}_u^X.$$

Then we extend this law on $\mathcal{G}_t^X$, as for X^{br} and we may check that $\bar{H}^{br}$ is a continuous process such that $\bar{H}_0^{br}=\bar{H}_t^{br}=0$, a.s. From this construction, the law of $\bar{H}^{br}$ is a regular version of the conditional law of $(\bar{H}_u, 0\le u\le T_s)$ given $T_s=t$.

Call $(e_u^{s,t},\, 0\le u\le s)$ the excursion process of $\bar{H}^{br}$. In particular

$$(e_u^{s,t},\, 0\le u\le s) \text{ has the same law as } (e_u,\, 0\le u\le s) \text{ given } T_s=t\,.$$

The following proposition is a straightforward consequence of the above definition and identity (4.3).

Proposition 4.2. *The law of the process* $\{(\mathcal{T}_{e_u^{s,t}}, d_{e_u^{s,t}}),\, 0\le v\le s\}$ *is a regular version of the law of the forest of size s,* $\mathcal{F}_{\bar{H}}^s$ *given* $\mathbf{M}_s=t$.

We will denote by $(\mathcal{F}_{\bar{H}}^{s,t}(u),\, 0\le u\le s)$ a process with values in $\mathbb{T}_c$ whose law under $\mathbb{P}$ is this of the Lévy forest of size s conditioned by $\mathbf{M}_s=t$, i.e. conditioned to have a mass equal to t.

In the remainder of this section, we will consider the case when the driving Lévy process is stable. We suppose that its index α belongs to $(1,2]$ so that condition (3.2) is satisfied. We will give a pathwise construction of the processes $(X^{br},\bar{H}^{br})$ from the path of the original processes $(X,\bar{H})$. This result leads to the following realization of the Lévy forest of size s conditioned by its mass. From now on, with no loss of generality, we suppose that $t=1$.

Theorem 4.3. *Assume that X is stable with index $\alpha\in(1,2]$ and define*

$$g=\sup\{u\le 1 : T_{u^{1/\alpha}}=s\cdot u\}.$$

(1) $\mathbb{P}$*-almost surely,*

$$0<g<1\,.$$

(2) *Under $\mathbb{P}$, the rescaled process*

$$(g^{(1-\alpha)/\alpha}\bar{H}(gu),\, 0\le u\le 1) \tag{4.9}$$

has the same law as $\bar{H}^{br}$ and is independent of g.

(3) *The forest $\mathcal{F}_{\bar{H}}^{s,1}$ of size s and mass 1 may be constructed from the rescaled process defined in (4.9), i.e. if we denote by* $u\mapsto\epsilon_u\overset{(def)}{=}(g^{(1-\alpha)/\alpha}e_u(gv),\, v\ge 0)$ *its process of excursions away from 0, then under $\mathbb{P}$,*

$$\mathcal{F}_{\bar{H}}^{s,1}\overset{(d)}{=}\{(\mathcal{T}_{\epsilon_u}, d_{\epsilon_u}),\, 0\le u\le s\}.$$

Proof: The process $T_u=\inf\{v : I_v\le -u\}$ is a stable subordinator with index $1/\alpha$. Therefore,

$$T_u<su^\alpha\,,\quad \text{i.o. as } u\downarrow 0 \quad\text{and}\quad T_u>su^\alpha\,,\quad \text{i.o. as } u\downarrow 0.$$

Indeed, if $u_n\downarrow 0$ then $\mathbb{P}(T_{u_n}<su_n^\alpha)=\mathbb{P}(T_1<s)>0$, so that $\mathbb{P}(\limsup_n\{T_{u_n}<su_n^\alpha\})\ge\mathbb{P}(T_1<s)>0$. But T satisfies Blumenthal 0-1 law, so this probability is 1. The same arguments prove that $\mathbb{P}(\limsup_n\{T_{u_n}>su_n^\alpha\})=1$ for any sequence $u_n\downarrow 0$. Since T has only positive jumps, we deduce that $T_u=su^\alpha$ infinitely often as u tends to 0, so we have proved the first part of the theorem.

The rest of the proof is a consequence of the following lemma.

Lemma 4.4. *The first passage bridge X^{br} enjoys the following path construction:*

$$X^{br}\overset{(d)}{=}(g^{-1/\alpha}X(gu),\, 0\le u\le 1)\,.$$

Moreover, the process $(g^{-1/\alpha}X(gu),\, 0\le u\le 1)$ is independent of g.

Proof: First note that for any $t>0$ the bivariate random variable (X_t, I_t) under $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure and there is a version of its density which is continuous. Indeed from the Markov property and (4.6), one has:

$$\begin{aligned}\mathbb{P}(I_t \le y \,|\, X_t = x) &= \mathbb{E}\left(\mathbf{1}_{\{T_y \le t\}} \frac{p_{t-T_y}(x-y)}{p_t(x)}\right) \\ &= \begin{cases} \int_0^t \frac{y}{s}\hat{p}_s(y)\frac{p_{t-s}(x-y)}{p_t(x)}\,ds\,, & \text{if } y \le x, \\ 1 & \text{if } y > x. \end{cases}\end{aligned}$$

Looking at the expressions of $\hat{p}_t(x)$ and $p_t(x)$ obtained from the Fourier inverse of the characteristic exponent of X and $-X$ respectively, we see that theses functions are continuously differentiable and that their derivatives are continuous in t. It allows us to conclude our proof.

Now let us consider the two dimensional self-similar strong Markov process $Y \stackrel{(\mathrm{def})}{=} (X, I)$ with state space $\{(u,v) \in \mathbb{R}^2 : v \le u\}$. From our preceding remark, the semigroup $q_t((x,y),(dx',dy')) = \mathbb{P}(X_t + x \in dx', y \wedge (I_t + x) \in dy')$ of Y is absolutely continuous with respect to a σ-finite measure and there is a version of its density which is continuous on the set $\{(u,v) \in \mathbb{R}^2 : v \le u\}$. Denote by $q_t((x,y),(x',y'))$ this version. We derive from (4.6), which may also be written as $t\mathbb{P}(-I_t \in ds) = s\hat{p}_t(s)\,ds$, that for all $-s \le x$,

$$q_t((x,y),(-s,-s)) = \mathbf{1}_{\{y \ge -s\}} \frac{1}{t}\hat{p}_t(s+x)\,. \tag{4.10}$$

Then we may apply a result due to Fitzsimmons et al. (1993) which asserts that the inhomogeneous Markov process on $[0,t]$, whose law is defined by

$$\mathbb{E}\left(F(Y_u, v \le u)\frac{q_{t-u}(Y_u,(x',y'))}{q_t((x,y),(x',y'))}\,\middle|\, Y_0 = (x,y)\right)\,, \quad 0 \le u < t\,, \tag{4.11}$$

where F is a measurable functional on $C([0,u],\mathbb{R}^2)$, is a regular version of the conditional law of $(Y_v, 0 \le v \le t)$ given $Y_t = (x',y')$, under $\mathbb{P}(\,\cdot\,|\,Y_0 = (x,y))$. This law is called the law of the bridge of Y from (x,y) to (x',y') with length t. Then from (4.10), the law which is defined in (4.11), when specifying it on the first coordinate and for $(x,y) = (0,0)$ and $(x',y') = (-s,-s)$, corresponds to the law of the first passage bridge which is defined in (4.4).

It remains to apply another result which may also be found in Fitzsimmons et al. (1993): observe that g is a backward time for Y in the sense of Fitzsimmons et al. (1993). Indeed we may check that $g = \sup\{u \le 1 : X_u = -su^{1/\alpha},\, X_u = I_u\}$, so that for all $u>0$, $\{g > u\} \in \sigma(Y_v : v \ge u)$. Then from Corollary 3 in Fitzsimmons et al. (1993), conditionally on g, the process $(Y_u, 0 \le u \le g)$ under $\mathbb{P}(\,\cdot\,|\,Y_0 = (0,0))$ has the law of a bridge from $(0,0)$ to Y_g with length g. (This result has been obtained and studied in a greater generality in Chaumont and Uribe, 2009.) But from the definition of g, we have $Y_g = (-sg^{1/\alpha}, -sg^{1/\alpha})$, so from the self-similarity of Y, under $\mathbb{P}$ the process

$$(g^{-1/\alpha}Y(g \cdot u)\,, 0 \le u \le 1)$$

has the law of the bridge of Y from $(0,0)$ to $(-s,-s)$ with length 1. The lemma follows by specifying this result on the first coordinate. $\square$

The second part of the theorem is a consequence of Lemma 4.4, the construction of $\bar{H}^{br}$ from X^{br} and the scaling property of $\bar{H}$. The third part follows from the definition of the conditioned forest $\mathcal{F}_{\bar{H}}^{s,1}$ in Proposition 4.3 and the second part of this theorem. $\square$

5. Invariance principles

We know from Lamperti (1967) that the only possible limits of sequences of re-scaled Galton-Watson processes are continuous state branching processes. Then a question which arises is: when can we say that the whole genealogy of the tree or the forest converges? In particular, do the height process, the contour process and the coding walk converge after a suitable re-scaling? This question has been completely solved by Duquesne and Le Gall (2002). Then one may ask the same for the trees or forests conditioned by their mass. In Duquesne (2003), the author proved that when the law ν is in the domain of attraction of a stable law, the height process, the contour process and the coding excursion of the corresponding Galton-Watson tree converge in law in the Skorokhod space of càdlàg paths. This work generalizes the main result in Aldous (1991) which concerns the Brownian case. In this section we will prove that in the stable case, an invariance principle also holds for sequences of Galton-Watson forests conditioned by their mass.

Recall from section 2 that for an offspring distribution μ we have set $\nu(k) = \mu(k+1)$, for $k = -1, 0, 1, \dots$. We make the following assumption:

$$\text{(H)} \quad \left\{ \begin{array}{l} \mu \text{ is aperiodic and there is an increasing sequence } (a_n)_{n\geq 0} \\ \text{such that } a_n \to +\infty \text{ and } S_n/a_n \text{ converges in law as } n \to +\infty \\ \text{toward the law of a non-degenerated r.v. } \theta. \end{array} \right.$$

Note that we are necessarily in the critical case, i.e. $\sum_k k\mu(k) = 1$, and that the law of θ is stable. Moreover, since $\nu(-\infty, -1) = 0$, the support of the Lévy measure of θ is $[0, \infty)$ and its index α is such that $1 < \alpha \leq 2$. Also (a_n) is a regularly varying sequence with index α. Under hypothesis (H), it has been proved by Grimvall (1974) that if Z is the Galton-Watson process associated to a tree or a forest with offspring distribution μ, then

$$\left(\frac{1}{a_n} Z_{[nt/a_n]},\, t \geq 0 \right) \Rightarrow (\overline{Z}_t,\, t \geq 0)\,, \quad \text{as } n \to +\infty,$$

where $(\overline{Z}_t,\, t \geq 0)$ is a continuous state branching process. Here and in the sequel, $\Rightarrow$ will stand for the weak convergence in the Skorokhod space of càdlàg trajectories. Recall from section 2 the definition of the discrete process (S, H, C). Under the same hypothesis, it is proved in Duquesne and Le Gall (2002), Corollary 2.5.1 that

$$\left[\left(\frac{1}{a_n} S_{[nt]}, \frac{a_n}{n} H_{[nt]}, \frac{a_n}{n} C_{2nt} \right),\, t \geq 0 \right] \Rightarrow \left[(X_t, \bar{H}_t, \bar{H}_t),\, t \geq 0) \right]\,, \quad \text{as } n \to +\infty, \tag{5.1}$$

where X is a stable Lévy process with law θ and $\bar{H}$ is the associated height process, as defined in section 3.

Again we fix a real $s > 0$ and we consider a sequence of positive integers (k_n) such that

$$\frac{k_n}{a_n} \to s\,, \quad \text{as } n \to +\infty. \tag{5.2}$$

Recall the notations of section 2. For any $n \ge 1$, let $(X^{br,n}, \bar{H}^{br,n}, C^{br,n})$ be the process whose law is this of

$$\left[\left(\frac{1}{a_n} S_{[nt]}, \frac{a_n}{n} H_{[nt]}, \frac{a_n}{n} C_{2nt}\right), 0 \le t \le 1\right],$$

under $\mathbb{Q}_\mu(\,\cdot \,|\, \zeta(\tau_1) + \cdots + \zeta(\tau_{k_n}) = n)$. We emphasize that since μ is aperiodic, $\mathbb{Q}_\mu(\zeta(\tau_1) + \cdots + \zeta(\tau_{k_n}) = n) > 0$, for all sufficiently large n. Note that we could also define this three dimensional process over the whole half-line $[0, \infty)$, rather than on $[0, 1]$. However, from the definitions in section 2, $\bar{H}^{br,n}$ and $C^{br,n}$ would simply vanish over $[1, \infty)$ and $X^{br,n}$ would be constant. The next theorem is the conditional version of the invariance principle that we have recalled in (5.1).

Theorem 5.1. *Under assumption* (H)*, the following weak convergence*

$$(X^{br,n}, \bar{H}^{br,n}, C^{br,n}) \Longrightarrow (X^{br}, \bar{H}^{br}, \bar{H}^{br})$$

holds on the space $\mathbb{D}^3$, as n tends to $+\infty$.

Remark 5.2. : By a classical time reversal argument, the weak convergence of the first coordinate in Theorem 5.1 implies the main result of Bryn-Jones and Doney (2006) and this of Caravenna and Chaumont (2008) in the spectrally positive case. Indeed, when X has no negative jumps, it is well known that the returned first passage bridge $(s + X^{br}_{1-u}, \, 0 \le u \le 1)$ is the bridge of a the dual process conditioned to stay positive from 0 to s with length 1. Similarly, the returned discrete first passage bridge whose law is this of $(k_n + S_{n-i}, \, 0 \le i \le n)$ under $\mathbb{P}(\,\cdot \,|\, T_{k_n} = n)$ has the same law as $(S_i, \, 0 \le i \le n)$ given $S_n = k_n$ and conditioned to stay positive. Then integrating with respect to the terminal values and applying Theorem 5.1 gives the result contained in Bryn-Jones and Doney (2006) and Caravenna and Chaumont (2008).

In order to give a sense to the convergence of the Lévy forest, we may consider the trees $\mathcal{T}^{br,n}$ and $\mathcal{T}^{br}$ which are coded respectively by the continuous processes with compact support, $C^{br,n}_u$ and $\bar{H}^{br}_u$, in the sense given at the beginning of section 3 (here we suppose that these processes are defined on $[1, \infty)$ and both equal to 0 on this interval). Roughly speaking the trees $\mathcal{T}^{br,n}$ and $\mathcal{T}^{br}$ are obtained from the original (conditioned) forests by rooting all the trees of these forests at a same root.

Corollary 5.3. *The sequence of trees $\mathcal{T}^{br,n}$ converges weakly in the space $\mathbb{T}_c$ endowed with the Gromov-Hausdorff topology towards $\mathcal{T}^{br}$.*

Proof: This results is a consequence of the weak convergence of the contour function $C^{br,n}$ toward $\bar{H}^{br}$ and the inequality

$$d_{GH}(\mathcal{T}_g, \mathcal{T}_{g'}) \le 2\|g - g'\|,$$

which is proved in Duquesne and Le Gall (2005), see Lemma 2.3. (We recall that d_{GH} the Gromov-Hausdorff distance which has been defined in Section 3.) □

A first step for the proof of Theorem 5.1 is to obtain the weak convergence of $(X^{br,n}, \bar{H}^{br,n})$ restricted to the Skorokhod space $\mathbb{D}([0, t])$ for any $t < 1$. Then we will derive the convergence on $\mathbb{D}([0, 1])$ from an argument of cyclic exchangeability. The convergence of the third coordinate $C^{br,n}$ is a consequence of its particular expression as a functional of the process $\bar{H}^{br,n}$. In the remainder of the proof, we suppose that S is defined on the same probability space as X and has step distribution ν under $\mathbb{P}$. Then H and C are given respectively by (2.1) and (2.2). Define also $T_k = \inf\{i : S_i = -k\}$, for all integers $k \geq 0$, so that the process $(X^{br,n}, \bar{H}^{br,n}, C^{br,n})$ has the same law as

$$\left[\left(\frac{1}{a_n}S_{[nt]}, \frac{a_n}{n}H_{[nt]}, \frac{a_n}{n}C_{2nt}\right), 0 \leq t \leq 1\right],$$

under the conditional probability $\mathbb{P}(\,\cdot\,|\,T_{k_n} = n)$.

Lemma 5.4. *For any $t < 1$, as n tends to $+\infty$, we have*

$$\left[(X_u^{br,n}, \bar{H}_u^{br,n}),\ 0 \leq u \leq t\right] \Longrightarrow \left[(X_u^{br}, \bar{H}_u^{br}),\ 0 \leq y \leq t\right] .$$

Proof: Let F be any bounded and continuous functional on $\mathbb{D}([0,t])$. By the Markov property at time $[nt]$ and identity (2.4),

$$\begin{aligned}
\mathbb{E}[F(X_u^{br,n}, \bar{H}_u^{br,n}; 0 \leq u \leq t)] &= \mathbb{E}\left[F\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; 0 \leq u \leq t\right) \,|\, T_{k_n} = n\right] \\
&= \mathbb{E}\left(\mathbf{1}_{\{[nt] \leq T_{k_n}\}} \frac{\mathbb{P}_{S_{[nt]}}(T_{k_n} = n - [nt])}{\mathbb{P}(T_{k_n} = n)} \times F\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; 0 \leq u \leq t\right)\right) \\
&= \mathbb{E}\left(\mathbf{1}_{\{\frac{1}{a_n}\underline{S}_{[nt]} \geq -\frac{k_n}{a_n}\}} \frac{n(k_n + S_{[nt]})}{k_n(n - [nt])} \frac{\mathbb{P}_{S_{[nt]}}(S_{n-[nt]} = -k_n)}{\mathbb{P}(S_n = -k_n)}\right. \\
&\qquad \left. \times F\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; 0 \leq u \leq t\right)\right),
\end{aligned} \tag{5.3}$$

where $\underline{S}_k = \inf_{i \leq k} S_i$. To simplify the computations in the remainder of this proof, we set $P^{(n)}$ for the law of the process $\left(\frac{1}{a_n}S_{[nu]}, \frac{a_n}{n}H_{[nu]}; u \geq 0\right)$ and P will stand for the law of the process $(X_u, \bar{H}_u; u \geq 0)$. Then $Y = (Y^1, Y^2)$ is the canonical process of the coordinates on the Skorokhod space $\mathbb{D}^2$ of càdlàg paths from $[0, \infty)$ into $\mathbb{R}^2$. We will also use special notations for the densities introduced in (4.4) and (5.3):

$$\begin{aligned}
D_t &= \mathbf{1}_{\{\underline{Y}_t^1 \geq -s\}} \frac{s + Y_t^1}{s(1-t)} \frac{\hat{p}_{1-t}(Y_t^1 + s)}{\hat{p}_1(s)}, \quad \text{and} \\
D_t^{(n)} &= \mathbf{1}_{\{\underline{Y}_{[nt]}^1 \geq -\frac{k_n}{a_n}\}} \frac{n(k_n + a_n Y_{[nt]}^1)}{k_n(n - [nt])} \frac{\mathbb{P}_{a_n Y_{[nt]}^1}(S_{n-[nt]} = -k_n)}{\mathbb{P}(S_n = -k_n)},
\end{aligned}$$

where $\underline{Y}_t^1 = \inf_{u \leq t} Y_u^1$. Put also F_t for $F(Y_u, 0 \leq u \leq t)$. To obtain our result, we have to prove that

$$\lim_{n \to +\infty} |E^{(n)}(F_t D_t^{(n)}) - E(F_t D_t)| = 0\,. \tag{5.4}$$

Let $M > 0$ and set $I_M(x) \stackrel{\text{(def)}}{=} \mathbf{1}_{[-s,M]}(x)$. By writing

$$E^{(n)}(F_t D_t^{(n)}) = E^{(n)}(F_t D_t^{(n)} I_M(Y_t^1)) + E^{(n)}(F_t D_t^{(n)}(1 - I_M(Y_t^1)))$$

and by doing the same for $E(F_t D_t)$, we have the following upper bound for the term in (5.4)

$$\begin{aligned}
|E^{(n)}(F_t D_t^{(n)}) - E(F_t D_t)| \quad &\leq \quad |E^{(n)}(F_t D_t^{(n)} I_M(Y_t^1)) - E(F_t D_t I_M(Y_t^1))| \\
&\quad + C E^{(n)}(D_t^{(n)}(1 - I_M(Y_t^1))) + C E(D_t(1 - I_M(Y_t^1)))\,,
\end{aligned}$$

where C is an upper bound for the functional F. But since D_t and $D_t^{(n)}$ are densities, $E^{(n)}(D_t^{(n)})=1$ and $E(D_t)=1$, hence

$$\begin{aligned}|E^{(n)}(F_tD_t^{(n)})-E(F_tD_t)| &\le |E^{(n)}(F_tD_t^{(n)}I_M(Y_t^1))-E(F_tD_tI_M(Y_t^1))| \quad (5.5)\\ &\quad +C[1-E^{(n)}(D_t^{(n)}I_M(Y_t^1))]+C[1-E(D_tI_M(Y_t^1))]\,.\end{aligned}$$

Now it remains to prove that the first term of the right hand side of the inequality (5.5) tends to 0, i.e.

$$|E^{(n)}(F_tD_t^{(n)}I_M(Y_t^1))-E(F_tD_tI_M(Y_t^1))|\to 0\,, \tag{5.6}$$

as $n\to+\infty$. Indeed, suppose that (5.6) holds, then by taking $F_t\equiv 1$, we see that the second term of the right hand side of (5.5) converges towards the third one. Moreover, $E(D_tI_M(Y_t^1))$ tends to 1 as M goes to $+\infty$. Therefore the second and the third terms in (5.5) tend to 0 as n and M go to $+\infty$.

Let us prove (5.6). From the triangle inequality and the expression of the densities D_t and $D_t^{(n)}$, we have

$$\begin{aligned}|E^{(n)}(F_tD_t^{(n)}I_M(Y_t^1))-E(F_tD_tI_M(Y_t^1))| &\le \sup_{x\in[-s,M]}|g_n(x)-g(x)|+\\ &\quad |E^{(n)}(F_tD_tI_M(Y_t^1))-E(F_tD_tI_M(Y_t^1))|\,, \quad (5.7)\end{aligned}$$

where $g_n(x)=\frac{n(k_n+x)}{k_n(n-[nt])}\frac{\mathbb{P}_x(S_{n-[nt]}=-k_n)}{\mathbb{P}(S_n=-k_n)}$ and $g(x)=\frac{s+x}{s(1-t)}\frac{p_{1-t}(x,-s)}{p_1(0,-s)}$. But thanks to Gnedenko local limit theorem, see Gnedenko and Kolmogorov (1954), Chap. 9, and the fact that $k_n/a_n\to s$, we have

$$\lim_{n\to+\infty}\sup_{x\in[-s,M]}|g_n(x)-g(x)|=0\,.$$

Moreover, recall that from Corollary 2.5.1 of Duquesne and Le Gall (2002),

$$P^{(n)}\Rightarrow P\,,$$

as $n\to+\infty$, where $\Rightarrow$ stands for the weak convergence of measures on $\mathbb{D}^2$. Finally, note that the discontinuity set of the functional $F_tD_tI_M(Y_t^1)$ is negligible for the probability measure P so that the last term in (5.7) tends to 0 as n goes to $+\infty$. □

Next we will prove the tightness of the sequence, $(X^{br,n},\bar H^{br,n})$. Define the height process associated with any downward skip free chain $x=(x_0,x_1,\dots,x_n,\dots)$, i.e. $x_0=0$ and $x_i-x_{i-1}\ge -1$, as follows:

$$H_n^x=\mathrm{card}\left\{i\in\{0,\dots,n-1\}:x_k=\inf_{i\le j\le n}x_j\right\}.$$

Let also $t(k)$ be the first passage time of x by $t(k)=\inf\{i:x_i=-k\}$ and for $n\ge k$, when $t(k)<\infty$, define the shifted chain:

$$\theta_{t(k)}(x)_i=\left\{\begin{array}{ll} x_{i+t(k)}+k, & \text{if } i\le n-t(k)\\ x_{t(k)+i-n}+x_n+k, & \text{is } n-t(k)\le i\le n\end{array}\right.\,,\qquad i=0,1,\dots,n\,,$$

which consists in inverting the pre-$t(k)$ and the post-$t(k)$ parts of x and sticking them together.

Lemma 5.5. *For any $k\ge 0$, we have almost surely*

$$H^{\theta_{t(k)}(x)}=\theta_{t(k)}(H^x)\,.$$

Proof: It is just a consequence of the fact that $t(k)$ is a zero of H^x. □

Lemma 5.6. *Let ν_n be a random variable which is uniformly distributed over $\{0,1,\dots,k_n\}$ and independent of S. Under $\mathbb{P}(\,\cdot\,|\,T_{k_n}=n)$, the first passage time T_{ν_n} is uniformly distributed over $\{0,1,\dots,n\}$.*

Proof: . It follows from elementary properties of random walks that for all $k\in\{0,1,\dots,k_n\}$, under $\mathbb{P}(\,\cdot\,|\,T_{k_n}=n)$, the chain $\theta_{T_k}(S)$ has the same law as $(S_i,\,0\le i\le n)$. As a consequence, for all $j\in\{0,1,\dots,n\}$,

$$\mathbb{P}(T_k=j\,|\,T_{k_n}=n)=\mathbb{P}(T_{k_n-k}=n-j\,|\,T_{k_n}=n)\,,$$

which allows us to conclude. □

Lemma 5.7. *The family of processes $(X^{br,n},\bar{H}^{br,n})$, $n\ge1$ is tight.*

Proof: Let $\mathbb{D}([0,t])$ be the Skorokhod space of càdlàg paths from $[0,t]$ to $\mathbb{R}$. In Lemma 5.4 we have proved the weak convergence of $(X^{br,n},\bar{H}^{br,n})$ restricted to the space $\mathbb{D}([0,t])$ for each $t>0$. Therefore, from Theorem 15.3 of Billingsley (1999), it suffices to prove that for all $\delta\in(0,1)$ and $\eta>0$,

$$\lim_{\delta\to0}\limsup_{n\to+\infty}\mathbb{P}\left(\sup_{s,t\in[1-\delta,1]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[1-\delta,1]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\right)=0\,. \tag{5.8}$$

From Lemma 5.6 the r.v. $V_n:=\inf\{t:X_t^{br,n}=-\frac{k}{n}\}$ is uniformly distributed over $\{0,1/n,\dots,1-1/n,1\}$ and we have for any $\varepsilon<1-\delta$,

$$\begin{aligned}\mathbb{P}\left(\sup_{s,t\in[1-\delta,1]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[1-\delta,1]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\right)\le\varepsilon+\delta+\\ \mathbb{P}\Bigg(V_n\in[\varepsilon,1-\delta],\ \sup_{s,t\in[1-\delta,1]}|X_t^{br,n}-X_s^{br,n}|>\eta,\\ \sup_{s,t\in[1-\delta,1]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\Bigg)\,.\end{aligned}$$

Now for a càdlàg path ω defined on $[0,1]$ and $t\in[0,1]$, define the shift:

$$\theta_t(\omega)_u=\begin{cases}\omega_{s+t}+u, & \text{if } s\le1-t\\ \omega_{t+u-1}+\omega_u+k, & \text{is } 1-t\le s\le1\end{cases}\ ,\qquad u\in[0,1]\,,$$

which consists in inverting the paths $(\omega_u,\,0\le u\le t)$ and $(\omega_u,\,t\le u\le1)$ and sticking them together. We can check on a picture the inclusion:

$$\begin{aligned}\{V_n\in[\varepsilon,1-\delta],\ \sup_{s,t\in[1-\delta,1]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[1-\delta,1]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\}\subset\\ \{\sup_{s,t\in[0,1-\varepsilon]}|\theta_{V_n}(X^{br,n})_t-\theta_{V_n}(X^{br,n})_s|>\eta,\\ \sup_{s,t\in[0,1-\varepsilon]}|\theta_{V_n}(\bar{H}^{br,n})_t-\theta_{V_n}(\bar{H}^{br,n})_s|>\eta\}\,.\end{aligned}$$

From Lemma 5.5 and the straightforward identity in law $X^{br,n}\stackrel{(d)}{=}\theta_{V_n}(X^{br,n})$, we deduce the two dimensional identity in law $(X^{br,n},\bar{H}^{br,n})\stackrel{(d)}{=}(\theta_{V_n}(X^{br,n}),$

$\theta_{V_n}(\bar{H}^{br,n})$). Hence from the above inequality and inclusion,

$$\mathbb{P}\left(\sup_{s,t\in[1-\delta,1]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[1-\delta,1]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\right)\le\varepsilon+\delta+$$
$$\mathbb{P}\left(\sup_{s,t\in[0,1-\varepsilon]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[0,1-\varepsilon]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\right).$$

But from Lemma 5.4 and Theorem 15.3 in Billingsley (1999), we have

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\mathbb{P}\left(\sup_{s,t\in[0,1-\varepsilon]}|X_t^{br,n}-X_s^{br,n}|>\eta,\ \sup_{s,t\in[0,1-\varepsilon]}|\bar{H}_t^{br,n}-\bar{H}_s^{br,n}|>\eta\right)=0\,.$$

which yields (5.8). □

Proof of Theorem 5.1: Lemma 5.4 shows that the sequence $(X^{br,n},\bar{H}^{br,n})$ converges toward $(X^{br},\bar{H}^{br})$ in the sense of finite dimensional distributions. Moreover tightness of this sequence has been proved in Lemma 5.7, so we conclude from Theorem 15.1 of Billingsley (1999). The convergence of the two first coordinates in Theorem 5.1 is proved, i.e. $(X^{br,n},\bar{H}^{br,n})\Longrightarrow(X^{br},\bar{H}^{br})$. Then we may deduce the functional convergence of the third coordinate from this convergence in law, using inequalities (2.3) and following similar arguments as in Theorem 2.4.1 of Duquesne and Le Gall (2002) or in Theorem 3.1 of Duquesne (2003). □

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