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The number of small blocks in exchangeable random partitions

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Abstract. Suppose Π is an exchangeable random partition of the positive integers and Π_n is its restriction to $\{1, \dots, n\}$. Let K_n denote the number of blocks of Π_n, and let $K_{n,r}$ denote the number of blocks of Π_n containing r integers. We show that if $0 < \alpha < 1$ and $K_n/(n^\alpha \ell(n))$ converges in probability to $\Gamma(1-\alpha)$, where ℓ is a slowly varying function, then $K_{n,r}/(n^\alpha \ell(n))$ converges in probability to $\alpha\Gamma(r-\alpha)/r!$. This result was previously known when the convergence of $K_n/(n^\alpha \ell(n))$ holds almost surely, but the result under the hypothesis of convergence in probability has significant implications for coalescent theory. We also show that a related conjecture for the case when K_n grows only slightly slower than n fails to be true.

1. Introduction

We begin by recalling some basic facts about exchangeable random partitions. Suppose π is a partition of the set $\mathbb{N}$ of positive integers. If σ is a permutation of $\mathbb{N}$, then we can define a partition $\sigma\pi$ such that the integers $\sigma(i)$ and $\sigma(j)$ are in the same block of $\sigma\pi$ if and only if i and j are in the same block of π. A random partition Π if $\mathbb{N}$ is said to be exchangeable if $\sigma\Pi$ and Π have the same distribution for all permutations σ of $\mathbb{N}$ having the property that $\sigma(j) = j$ for all but finitely many j.

Kingman (1978) proved an analog of de Finetti's Theorem that characterizes all possible exchangeable random partitions. He showed that there is a one-to-one correspondence between distributions of exchangeable random partitions and probability measures on the infinite simplex $\Delta = \{(x_i)_{i=1}^\infty : x_1 \geq x_2 \geq \dots \geq 0, \sum_{i=1}^\infty x_i \leq 1\}$. Given a probability distribution μ on Δ, the associated exchangeable random partition is constructed as follows. First, choose a random sequence $(P_j)_{j=1}^\infty$ with distribution μ. Then define random variables $(\xi_k)_{k=1}^\infty$ that

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are conditionally independent given $(P_j)_{j=1}^\infty$ and satisfy $P(\xi_k = i | (P_j)_{j=1}^\infty) = P_i$ and $P(\xi_k = -k | (P_j)_{j=1}^\infty) = 1 - \sum_{j=1}^\infty P_j$. Finally, define Π to be the random partition of $\mathbb{N}$ such that two integers i and j are in the same block of Π if and only if $\xi_i = \xi_j$.

It follows from this construction and the Law of Large Numbers that if B is a block of an exchangeable random partition Π, then the asymptotic frequency of the block, defined by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{i \in B\}},$$

exists almost surely. The nonzero asymptotic frequencies of the blocks of Π are the nonzero terms of the sequence $(P_j)_{j=1}^\infty$. Each integer is in a block having positive asymptotic frequency with probability $\sum_{j=1}^\infty P_j$ and is in a singleton block with probability $1 - \sum_{j=1}^\infty P_j$.

Given an exchangeable random partition Π of $\mathbb{N}$, let Π_n denote its restriction to $\{1, \dots, n\}$. That is, Π_n is the partition of $\{1, \dots, n\}$ such that two integers i and j in $\{1, \dots, n\}$ are in the same block of Π_n if and only if they are in the same block of Π. Let K_n be the number of blocks of Π_n, and let $K_{n,r}$ be the number of blocks of Π_n having size r. In this paper, we show how asymptotic results for the random variables $K_{n,r}$ as $n \to \infty$ can be deduced from the asymptotic behavior of K_n. Such results have already been proved, and are summarized in Gnedin et al. (2007), for the case in which the asymptotic frequencies P_j are deterministic and sum to one. This is the setting of the classical infinite occupancy problem, in which infinitely many balls are placed independently into infinitely many boxes, with each ball going into the jth box with probability P_j. Here we extend these results to the general case of random P_j and explore the applications of this extension to coalescent theory and population genetics.

We note that in addition to the results below concerning the asymptotic behavior of $K_{n,r}$, Central Limit Theorems have been established for the number of small blocks in exchangeable random partitions under certain conditions. See Karlin (1967) for some early work in this direction and Barbour and Gnedin (2009) for some recent extensions.

1.1. *The power law case.* We first consider the case in which the number of blocks K_n grows like n^α, where $0 < \alpha < 1$. The proposition below is essentially due to Karlin (1967). More precisely, it follows from combining Theorem 1 of Karlin (1967) with a Tauberian theorem. The result also appears as Corollary 21 in the recent survey of Gnedin et al. (2007). Recall that a measurable function $\ell : (0, \infty) \to (0, \infty)$ is said to be slowly varying if for all $c > 0$, we have $\lim_{y\to\infty} \ell(cy)/\ell(y) = 1$.

Proposition 1.1. *Let $(p_j)_{j=1}^\infty$ be a deterministic sequence such that $p_1 \geq p_2 \geq \dots \geq 0$ and $\sum_{j=1}^\infty p_j = 1$. For $x > 0$, let $g(x) = \max\{j : p_j \geq x\}$. Let Π be an exchangeable random partition of $\mathbb{N}$ whose asymptotic block frequencies are given by $(p_j)_{j=1}^\infty$ almost surely, and define K_n and $K_{n,r}$ as above. Suppose $0 < \alpha < 1$. Suppose $\ell : (0, \infty) \to (0, \infty)$ is a slowly varying function. We have*

$$\lim_{x\to 0} \frac{x^\alpha g(x)}{\ell(1/x)} = 1 \qquad (1.1)$$

if and only if

$$\lim_{n\to\infty} \frac{K_n}{n^\alpha \ell(n)} = \Gamma(1 - \alpha) \quad \text{a.s.} \qquad (1.2)$$

These two statements imply that for all $r \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{K_{n,r}}{n^\alpha \ell(n)} = \frac{\alpha\Gamma(r-\alpha)}{r!} \quad \text{a.s.} \tag{1.3}$$

Our main theorem is an extension of Proposition 1.1 to general exchangeable random partitions. It is an immediate consequence of Proposition 1.1 that even when the P_j may be random, the condition (1.2) implies (1.3). The result below says that this implication remains valid even when the convergence in (1.2) holds only in probability. As we will see shortly, this result has applications in coalescent theory, where it can be much easier to establish convergence in probability for K_n than almost sure convergence.

Theorem 1.2. *Suppose Π is an exchangeable random partition of $\mathbb{N}$, and define K_n and $K_{n,r}$ as above. Suppose $0 < \alpha < 1$, and suppose $\ell : (0,\infty) \to (0,\infty)$ is a slowly varying function. If*

$$\lim_{n\to\infty} \frac{K_n}{n^\alpha \ell(n)} = \Gamma(1-\alpha) \quad \text{in probability} \tag{1.4}$$

then for all $r \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{K_{n,r}}{n^\alpha \ell(n)} = \frac{\alpha\Gamma(r-\alpha)}{r!} \quad \text{in probability.}$$

We prove Theorem 1.2 in Section 2. It will follow from this proof (see Lemma 2.6 below) that (1.4) implies that the limit (1.1) holds in probability. However, the converse implication is false. Of course, it is clear that the converse can not hold for general exchangeable random partitions because (1.1) can hold even when $\sum_{j=1}^\infty p_j < 1$, in which case K_n will be of order n rather than of order n^α. However, as the next example shows, even under the additional condition that $\sum_{j=1}^\infty P_j = 1$, it is possible for the limit (1.1) to hold in probability but for (1.4) to fail.

Example 1.3. There exists an exchangeable random partition Π of $\mathbb{N}$ whose asymptotic frequencies satisfy $\sum_{j=1}^\infty P_j = 1$ a.s. such that if $G(x) = \max\{j : P_j \geq x\}$, then

$$\lim_{x\to 0} x^\alpha G(x) = 1 \quad \text{in probability}$$

but $n^{-\alpha} K_n$ does not converge in probability to $\Gamma(1-\alpha)$ as $n \to \infty$.

We describe the example in detail, and prove that it has the stated properties, in Section 3.

1.2. *The case in which K_n is only slightly smaller than n.* Proposition 1.1 and Theorem 1.2 give asymptotic results for $K_{n,r}$ when K_n grows like n^α for $0 < \alpha < 1$. The result below concerns the case when K_n grows just slightly slower than n. This result can be obtained from results in Gnedin et al. (2007) by combining Propositions 14 and 18 with Lemma 1, Proposition 2, and the remarks before and after Proposition 2.

Proposition 1.4. *Let $(p_j)_{j=1}^\infty$ be a deterministic sequence such that $p_1 \geq p_2 \geq \cdots \geq 0$ and $\sum_{j=1}^\infty p_j = 1$. For $x > 0$, let $g(x) = \max\{j : p_j \geq x\}$. Let Π be an exchangeable random partition of $\mathbb{N}$ whose asymptotic block frequencies are given by $(p_j)_{j=1}^\infty$ almost surely, and define K_n and $K_{n,r}$ as above. Suppose $\ell : (0,\infty) \to$*

$(0,\infty)$ is a slowly varying function, and for $t > 0$, let $\ell_1(t) = \int_t^\infty \ell(s)/s\,ds$. Suppose that

$$\lim_{x \to 0} \frac{xg(x)}{\ell(1/x)} = 1. \tag{1.5}$$

Then

$$\lim_{n \to \infty} \frac{K_n}{n\ell_1(n)} = \lim_{n \to \infty} \frac{K_{n,1}}{n\ell_1(n)} = 1 \quad \text{a.s.} \tag{1.6}$$

Also, for integers $r \geq 2$,

$$\lim_{n \to \infty} \frac{K_{n,r}}{n\ell(n)} = \frac{1}{r(r-1)} \qquad \text{a.s.} \tag{1.7}$$

Our next result addresses a question that is left open by Proposition 1.4. Although (1.5) implies (1.6) and (1.7), one can also ask whether there is a result parallel to Theorem 1.2 in which we obtain asymptotic results for $K_{n,r}$ just from the asymptotics of K_n. However, the example below, which we describe in detail in Section 4, shows that the condition $K_n/(n\ell_1(n)) \to 1$ a.s. is not sufficient to imply that the convergence in (1.7) holds, even in probability. Note that in the notation of Proposition 1.4, if $\ell(t) = (\log t)^{-2}$ for all $t \geq T > 1$, then $\ell_1(t) = (\log t)^{-1}$ for all $t \geq T$.

Example 1.5. There exists an exchangeable random partition Π of $\mathbb{N}$ such that if K_n and $K_{n,r}$ are defined as above, then

$$\lim_{n \to \infty} \frac{(\log n)K_n}{n} = 1 \quad \text{a.s.},$$

but for all integers $r \geq 2$, the quantity $n^{-1}(\log n)^2 K_{r,n}$ does not converge in probability to $1/[r(r-1)]$ as $n \to \infty$.

1.3. *Applications to coalescent theory and population genetics.* At first glance, Theorem 1.2 may appear to be only a very minor technical improvement over Proposition 1.1. However, Theorem 1.2 has significant implications for coalescent theory, where it can be much easier to prove convergence in probability and establish (1.4) than to prove the almost sure convergence needed to obtain (1.2).

Suppose we take a sample of size n from a population and follow the ancestral lines of the sampled individuals backwards in time. The ancestral lines will coalesce until all of the sampled individuals are traced back to a single common ancestor. This process can be modeled by a stochastic process taking its values in the set of partitions of $\{1, \dots, n\}$. The standard coalescent model is Kingman's coalescent, introduced by Kingman (1982), in which it is assumed that only two lineages ever merge at a time and each transition that involves the merging of two lineages happens at rate one. This means that when there are b lineages, the amount of time before the next merger has an exponential distribution with rate $\binom{b}{2}$.

Within the last decade, there has been considerable interest in alternative models of coalescence, called coalescents with multiple mergers or Λ-coalescents, that allow many ancestral lines to merge at once. Such processes were introduced by Pitman (1999) and Sagitov (1999). If Λ is a finite measure on $[0,1]$, then the Λ-coalescent is the coalescent process having the property that whenever there are b lineages, each transition that involves k lineages merging into one happens at rate

$$\lambda_{b,k} = \int_0^1 x^{k-2}(1-x)^{b-k}\,\Lambda(dx).$$

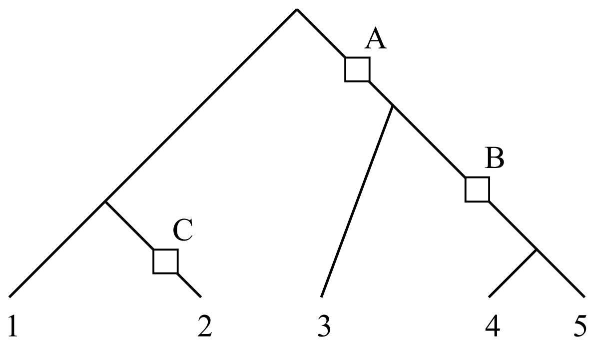

FIGURE 1.1. This figure shows the genealogy of five sampled individuals. The boxes represent mutations. Individual 1 inherited no mutations, individual 2 inherited mutation C, individual 3 inherited mutation A, and individuals 4 and 5 inherited mutations A and B. Therefore, the allelic partition is $\Pi_5 = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$. We have $K_5 = 4$. Also, $K_{5,1} = 3$, $K_{5,2} = 1$, and $K_{5,3} = K_{5,4} = K_{5,5} = 0$.

Multiple mergers of ancestral lines could arise in populations with large family sizes, as many ancestral lines could be traced back to the individual that had a large number of offspring. They could also arise as a result of natural selection because many ancestral lines could get traced back to an individual that had a beneficial mutation which spread rapidly to a large fraction of the population.

To model mutations, we put marks representing mutations at points of a rate θ Poisson process along each branch of the coalescent tree. One can then define a random partition Π_n of $\{1, \ldots, n\}$, often called the allelic partition, by declaring i and j to be in the same block of Π_n if and only if the ith and jth sampled individuals inherit the same mutations. These partitions Π_n can be defined consistently as n increases simply by sampling more individuals, so by Kolmogorov's Extension Theorem, on some probability space there is an exchangeable random partition Π of $\mathbb{N}$ such that Π_n is the restriction to Π of $\{1, \ldots, n\}$. When the underlying coalescent process is Kingman's coalescent, the distribution of Π_n is given by the Ewens Sampling Formula, which was introduced in Ewens (1972). The probability that Π_n has a_j blocks of size j for $j = 1, \ldots, n$ is given by

$$\frac{n!}{2\theta(2\theta+1)\ldots(2\theta+n-1)} \prod_{j=1}^{n} \left(\frac{2\theta}{j}\right)^{a_j} \frac{1}{a_j!}.$$

When the underlying coalescent process is some other Λ-coalescent, there is no simple expression for the distribution of Π. However, defining K_n and $K_{n,r}$ from Π_n as above, it was shown in Berestycki et al. (2007, 2008) that if Λ is the Beta$(\alpha, 2-\alpha)$ distribution with $0 < \alpha < 1$, then

$$\lim_{n\to\infty} \frac{K_n}{n^\alpha} = \frac{\theta(2-\alpha)(1-\alpha)\Gamma(2-\alpha)}{\alpha} \quad \text{in probability.} \tag{1.8}$$

It was then shown in Berestycki et al. (2007) that

$$\lim_{n\to\infty} \frac{K_{n,r}}{n^\alpha} = \frac{\theta(2-\alpha)(1-\alpha)^2\Gamma(r-\alpha)}{r!} \quad \text{in probability.} \tag{1.9}$$

Note that α here corresponds to $2-\alpha$ in Berestycki et al. (2007, 2008). The proof of (1.9) in Berestycki et al. (2007) is rather technical, exploiting a connection between beta coalescents and the genealogy of continuous-state branching processes. However, Theorem 1.2 makes it possible to deduce (1.9) immediately from (1.8). We also note that the convergence in (1.8) was later shown in Berestycki et al. (2010) to hold almost surely, allowing (1.9) to be established via Proposition 1.1. On the other hand, if Λ is the uniform distribution on $[0,1]$, corresponding to $\alpha = 1$ above, it was shown in Basdevant and Goldschmidt (2008), building on work of Drmota et al. (2007), that

$$\lim_{n\to\infty} \frac{(\log n)K_n}{n} = \theta \quad \text{in probability.} \tag{1.10}$$

It was also shown in Basdevant and Goldschmidt (2008) that

$$\lim_{n\to\infty} \frac{(\log n)^2 K_{n,r}}{n} = \frac{\theta}{k(k-1)} \quad \text{in probability.} \tag{1.11}$$

However, Example 1.5 establishes that (1.10) does not imply (1.11). Indeed, the proof of (1.11) in Basdevant and Goldschmidt (2008) involves a detailed analysis of a Markov chain on different time scales.

1.4. *A model of a growing population.* To illustrate another application of Theorem 1.2, we consider the following model of a population that grows in size over time. Fix $\gamma > 0$ and a positive integer N. Assume that for each positive integer k, there are $\lceil Nk^{-\gamma} \rceil$ individuals in generation $-k$. For simplicity, assume that the number of individuals in generation zero is the same as the number of individuals in generation -1, so there are N individuals in generations 0 and 1 but fewer in earlier generations. To give the model a genealogical structure, we assume, as in the standard Wright-Fisher model, that each individual chooses its parent uniformly at random from the individuals in the previous generation.

Now sample n individuals from the population at time zero, and follow their ancestral lines backwards in time. We can represent the genealogy of these sampled individuals by a coalescent process $(\Psi_{N,n}(t), t \geq 0)$ taking its values in the set of partitions of $\{1, \dots, n\}$, where two integers i and j are in the same block of the partition $\Psi_{N,n}(t)$ if and only if the ith and jth individuals in the sample have the same ancestor at time $-\lfloor N^{1/(1+\gamma)} t \rfloor$. It is easy to check that as $N \to \infty$, these processes converge to a coalescent process $(\Psi_n(t), t \geq 0)$ having the property that at time t, two lineages (that is, two blocks of the partition) are merging at rate t^γ. To see this, note that in generation $N^{1/(1+\gamma)}t$, two individuals have the same ancestor with probability approximately $N^{-1}(N^{1/(1+\gamma)}t)^\gamma$, and multiplying this expression by the time-scaling factor $N^{1/(1+\gamma)}$ gives the coalescence rate of t^γ. Note that $(\Psi_n(t), t \geq 0)$ is a time-inhomogeneous Markov chain.

The process $(\Psi_n(t), t \geq 0)$ can be obtained as a time-change of Kingman's coalescent. Indeed, let $(\Theta_n(t), t \geq 0)$ be Kingman's coalescent started with n lineages. That is $(\Theta_n(t), t \geq 0)$ is a continuous-time, time-homogeneous Markov chain taking values in the set of partitions of $\{1, \dots, n\}$ such that $\Theta_n(0) = \{\{1\}, \{2\}, \dots, \{n\}\}$, each transition that involves merging two blocks of the partition happens at rate one, and no other transitions are possible. Then we can define

$$\Psi_n(t) = \Theta_n\left(\frac{t^{\gamma+1}}{\gamma+1}\right). \tag{1.12}$$

The time change makes Ψ_n a time-inhomogeneous Markov chain in which at time t, each pair of blocks is merging at rate t^γ.

We will now work with the coalescent process $(\Psi_n(t), t \geq 0)$ and, as before, put mutations along each lineage at times of a rate θ Poisson process. Then define the partition Π_n such that i and j are in the same block of Π_n if and only if the ith and jth sampled individuals inherit the same mutations. The partitions Π_n can be defined consistently as n varies, so there is an exchangeable random partition Π of $\mathbb{N}$ such that Π_n is the restriction of Π to $\{1, \dots, n\}$. Define K_n and $K_{n,r}$ as before. We obtain the following result.

Theorem 1.6. *Consider the time-inhomogeneous coalescent process with mutations described above. Let $\alpha = \gamma/(1+\gamma) \in (0,1)$. We have*

$$\lim_{n\to\infty} \frac{K_n}{n^\alpha} = \frac{\theta 2^{1-\alpha}(1-\alpha)^\alpha \pi}{\sin(\pi\alpha)} \quad \text{in probability} \tag{1.13}$$

and for all $r \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{K_{n,r}}{n^\alpha} = \frac{\theta 2^{1-\alpha}(1-\alpha)^\alpha \pi}{\sin(\pi\alpha)} \cdot \frac{\alpha\Gamma(r-\alpha)}{r!\Gamma(1-\alpha)} \quad \text{in probability.} \tag{1.14}$$

Of course, in view of Theorem 1.2, equation (1.14) follows immediately from (1.13), so we need only prove (1.13), which we do in Section 5.

Note that for both the beta coalescent and for the time-inhomogeneous coalescent described above, we have

$$\lim_{n\to\infty} \frac{K_{n,r}}{K_n} = \frac{\alpha\Gamma(r-\alpha)}{r!\Gamma(1-\alpha)} \quad \text{in probability.} \tag{1.15}$$

The left-hand side of (1.15) is the fraction of blocks of the allelic partition having size r, and the sequence of numbers $K_{n,r}$ for $1 \leq r \leq n$ is often called the allele frequency spectrum. Thus, (1.15) says that we get the same allele frequency spectrum for these two models, as we would with any coalescent model having the property that K_n grows like n^α.

One of the central goals of population genetics is to use information about a sample from a current population to obtain information about the history of the population. Distinguishing among various factors that could cause the genealogy of the population to differ from Kingman's coalescent can be challenging. See, for example, Jense et al. (2005) and Ometto et al. (2005) for a discussion of the issue of distinguishing the effects of natural selection from demographic factors such as changing population size. Therefore, from the perspective of population genetics, Theorem 1.6 is perhaps disappointing. Theorem 1.6 shows that the allele frequency spectrum that arises when the genealogy is given by a beta coalescent, as could be the case for populations with large family sizes, could also arise in a population whose size is increasing over time. Thus, one can not necessarily use the allele frequency spectrum to distinguish populations with large family sizes from populations that are increasing in size. In general, Proposition 1.1 and Theorem 1.2 suggest that the same allele frequency spectrum may arise in a wide variety of models, and thus may explain part of the difficulty in distinguishing among various factors that could cause the genealogy of a population to differ from Kingman's coalescent.

2. Proof of Theorem 1.2

Throughout this section, we assume that $0 < \alpha < 1$ and that Π is an exchangeable random partition. We define K_n and $K_{n,r}$ as in Theorem 1.2. We assume that $\ell : (0,\infty) \to (0,\infty)$ is a slowly varying function and that (1.4) holds. We denote by $P_1 \geq P_2 \geq \dots$ the asymptotic frequencies of the blocks of Π. Note that (1.4) implies that $\sum_{j=1}^{\infty} P_j = 1$ a.s. because $\liminf_{n\to\infty} n^{-1}K_n > 0$ almost surely on the event that $\sum_{j=1}^{\infty} P_j < 1$. For $x > 0$, define $G(x) = \max\{j : P_j \geq x\}$, which is a random variable because the P_j are random.

At times in the proof of Theorem 1.2, it will be useful to use a technique called Poissonization. Let $(N(t), t \geq 0)$ be a rate one Poisson process, so that $N(t)$ has the Poisson distribution with mean t for all t. Define the random variable

$$\Phi(t) = E[K_{N(t)} | (P_j)_{j=1}^{\infty}].$$

Likewise, for positive integers r, define

$$\Phi_r(t) = E[K_{N(t),r} | (P_j)_{j=1}^{\infty}].$$

We have (see the proof of Proposition 17 in Gnedin et al. (2007)),

$$\Phi(t) = t \int_0^{\infty} e^{-tx} G(x)\, dx \quad \text{a.s.} \tag{2.1}$$

Also,

$$\Phi_r(t) = \frac{t^r}{r!} \sum_{j=1}^{\infty} P_j^r e^{-tP_j} \quad \text{a.s.} \tag{2.2}$$

By conditioning on $(P_j)_{j=1}^{\infty}$ and applying Lemma 1 and Proposition 2 of Gnedin et al. (2007), we get

$$\lim_{n\to\infty} \frac{K_n}{\Phi(n)} = 1 \quad \text{a.s.} \tag{2.3}$$

Using the remarks following Proposition 2 of Gnedin et al. (2007), we have for all positive integers r,

$$\lim_{n\to\infty} \frac{\sum_{s=r}^{\infty} K_{n,s}}{\sum_{s=r}^{\infty} \Phi_s(n)} = 1 \quad \text{a.s.} \tag{2.4}$$

Lemma 2.1 below, known as Potter's Theorem, is Theorem 1.5.6(i) of Bingham et al. (1987) and gives some bounds on slowly varying functions. Note that since Theorem 1.2 only concerns the values of $\ell(n)$ for $n \in \mathbb{N}$, we may and will assume, here and throughout this section, that ℓ is bounded away from zero and infinity on $(0, x]$ for any $x > 0$.

Lemma 2.1. *Suppose $\ell : (0,\infty) \to (0,\infty)$ is a slowly varying function. Let $\delta > 0$. There exists a positive number $x_0(\delta)$ such that if $x \geq x_0(\delta)$ and $\lambda \geq 1$, then*

$$\frac{1}{(1+\delta)\lambda^{\delta}} \leq \frac{\ell(\lambda x)}{\ell(x)} \leq (1+\delta)\lambda^{\delta}. \tag{2.5}$$

Also, there exists a constant $C > 0$ such that $\ell(x) \geq Cx^{-\delta}$ for all $x \geq x_0(\delta)$.

Lemma 2.2. *We have*

$$\lim_{t\to\infty} \frac{t^{1-\alpha}}{\ell(t)} \int_0^{\infty} e^{-tx} G(x)\, dx = \Gamma(1-\alpha) \quad \text{in probability.}$$

Proof: We use Poissonization. Combining (1.4) and (2.3), we get

$$\lim_{n\to\infty} \frac{\Phi(n)}{n^\alpha \ell(n)} = \Gamma(1-\alpha) \quad \text{in probability.}$$

Since $t \mapsto \Phi(t)$ is nondecreasing and ℓ is slowly varying, it follows from Lemma 2.1 that

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^\alpha \ell(t)} = \Gamma(1-\alpha) \quad \text{in probability.}$$

The result now follows from (2.1). □

Lemma 2.3. *We have*

$$\lim_{t\to\infty} \frac{t^{1-\alpha}}{\ell(t)} \int_0^\infty e^{-tx}(G(x) - x^{-\alpha}\ell(1/x))\,dx = 0 \quad \text{in probability.}$$

Proof: In view of Lemma 2.2, it suffices to show that

$$\lim_{t\to\infty} \frac{t^{1-\alpha}}{\ell(t)} \int_0^\infty e^{-tx} x^{-\alpha}\ell(1/x)\,dx = \Gamma(1-\alpha). \tag{2.6}$$

Choose δ such that $\delta + \alpha < 1$, and choose x_0 so that (2.5) holds for $x \geq x_0$ and $\lambda \geq 1$. Substituting $y = tx$, we get

$$\begin{aligned}\frac{t^{1-\alpha}}{\ell(t)} \int_0^\infty e^{-tx}x^{-\alpha}\ell(1/x)\,dx &= \frac{1}{\ell(t)} \int_0^\infty e^{-y}y^{-\alpha}\ell(t/y)\,dy \\ &= \int_0^\infty e^{-y}y^{-\alpha}\left(\frac{\ell(t/y)}{\ell(t)}\right)\mathbf{1}_{\{y\leq t/x_0\}}\,dy + \frac{1}{\ell(t)}\int_{t/x_0}^\infty e^{-y}y^{-\alpha}\ell(t/y)\,dy. \end{aligned} \tag{2.7}$$

By (2.5), we have $\ell(t/y)/\ell(t) \leq \max\{2y^{-\delta}, 2y^{\delta}\}$ whenever $t \geq x_0$ and $0 < y \leq t/x_0$. Also, since ℓ is a slowly varying function, $\lim_{t\to\infty} \ell(t/y)/\ell(t) = 1$ for all $y > 0$. Therefore, by the Dominated Convergence Theorem,

$$\lim_{t\to\infty} \int_0^\infty e^{-y}y^{-\alpha}\left(\frac{\ell(t/y)}{\ell(t)}\right)\mathbf{1}_{\{y\leq t/x_0\}}\,dy = \int_0^\infty e^{-y}y^{-\alpha}\,dy = \Gamma(1-\alpha). \tag{2.8}$$

Recall from Lemma 2.1 that there is a constant C such that $\ell(t) \geq Ct^{-\delta}$ for all $t \geq x_0$. By assumption, there is a constant B such that $\ell(x) \leq B$ for $0 < x \leq x_0$. Therefore,

$$\limsup_{t\to\infty} \frac{1}{\ell(t)} \int_{t/x_0}^\infty e^{-y}y^{-\alpha}\ell(t/y)\,dy \leq \limsup_{t\to\infty} \frac{Bt^\delta}{C}\left(\frac{t}{x_0}\right)^{-\alpha} e^{-t/x_0} = 0. \tag{2.9}$$

Equation (2.6) follows from (2.7), (2.8), and (2.9). □

Lemma 2.4. *There exists a positive number C_0 such that if $C > C_0$, then*

$$\lim_{x\to 0} P(G(x) > C\ell(1/x)x^{-\alpha}) = 0.$$

Proof: Suppose $G(x) \geq C\ell(1/x)x^{-\alpha}$. Since $x \mapsto G(x)$ is nonincreasing,

$$t^{1-\alpha}\int_0^\infty e^{-ty}G(y)\,dy \geq t^{1-\alpha}\int_0^x e^{-ty}C\ell(1/x)x^{-\alpha}\,dy = C(tx)^{-\alpha}\ell(1/x)(1-e^{-tx}).$$

Therefore, if $t = 1/x$, then

$$\frac{t^{1-\alpha}}{\ell(t)}\int_0^\infty e^{-ty}G(y)\,dy \geq C(1-e^{-1}).$$

By Lemma 2.2, the result follows with $C_0 = \Gamma(1-\alpha)/(1-e^{-1})$. □

Lemma 2.5. *There exists a positive number C such that if*

$$\bar{G}(x) = G(x)\mathbf{1}_{\{G(x) > C\ell(1/x)x^{-\alpha}\}},$$

then

$$\lim_{t\to\infty} \frac{t^{1-\alpha}}{\ell(t)} \int_0^\infty e^{-tx}\bar{G}(x)\,dx = 0 \ \text{ in probability.}$$

Proof: Let $\epsilon > 0$. Choose $C_1 > C_0$, where C_0 is the constant from Lemma 2.4, and let $C = 2^{1+\alpha}C_1$. Choose an integer M large enough that $C_1 2^{-M\alpha}e^{-2^M} < \epsilon/2$ and $2^{-M(1-\alpha)}\Gamma(1-\alpha)e < \epsilon/4$. For $t > 0$, define the event

$$A_t = \{G(2^k t^{-1}) \leq C_1\ell(2^{-k}t)(2^k t^{-1})^{-\alpha} \text{ for } k = -M, -M+1, \dots, M-1, M\}.$$

By Lemma 2.4, there exists $T_1 < \infty$ such that if $t > T_1$, then $P(A_t) > 1 - \epsilon/2$. Because $\bar{G}(x) \leq G(x) \leq G(2^M t^{-1})$ for all $x \geq 2^M t^{-1}$, on the event A_t we have

$$\begin{aligned}\frac{t^{1-\alpha}}{\ell(t)} \int_{2^M t^{-1}}^\infty e^{-tx}\bar{G}(x)\,dx &\leq \frac{t^{1-\alpha}}{\ell(t)} \cdot C_1\ell(2^{-M}t)(2^M t^{-1})^{-\alpha} \int_{2^M t^{-1}}^\infty e^{-tx}\,dx \\ &= C_1 2^{-M\alpha}e^{-2^M} \cdot \frac{\ell(2^{-M}t)}{\ell(t)} < \frac{\ell(2^{-M}t)}{\ell(t)} \cdot \frac{\epsilon}{2}. \qquad (2.10)\end{aligned}$$

Because ℓ is slowly varying, $\ell(2^{-M}t)/\ell(t) \to 1$ as $t \to \infty$. Therefore, there exists a T_2 such that for $t > T_2$, on A_t we have

$$\frac{t^{1-\alpha}}{\ell(t)} \int_{2^M t^{-1}}^\infty e^{-tx}\bar{G}(x)\,dx < \frac{\epsilon}{2}. \qquad (2.11)$$

Also, on A_t, if $2^k t^{-1} \leq x \leq 2^{k+1}t^{-1}$ for some k satisfying $-M \leq k \leq M-1$, then

$$G(x) \leq G(2^k t^{-1}) \leq C_1\ell(2^{-k}t)(2^k t^{-1})^{-\alpha} \leq C_1\ell(2^{-k}t)(x/2)^{-\alpha} = 2^\alpha C_1\ell(2^{-k}t)x^{-\alpha}.$$

By Lemma 2.1, there exists $T_3 < \infty$ such that if $t > T_3$ and $2^k t^{-1} \leq x \leq 2^{k+1}t^{-1}$ for some integer k satisfying $-M \leq k \leq M-1$, then $\ell(2^{-k}t)/\ell(1/x) \leq 2$. Therefore, if A_t occurs and $t > T_3$ then

$$G(x) \leq 2^{1+\alpha}C_1\ell(1/x)x^{-\alpha} = C\ell(1/x)x^{-\alpha}.$$

In this case, $\bar{G}(x) = 0$ for $2^{-M}t^{-1} \leq x \leq 2^M t^{-1}$ and thus

$$\frac{t^{1-\alpha}}{\ell(t)} \int_{2^{-M}t^{-1}}^{2^M t^{-1}} e^{-tx}\bar{G}(x)\,dx = 0. \qquad (2.12)$$

If $0 \leq x \leq 2^{-M}t^{-1}$, then $e^{-tx} \leq 1 \leq e \cdot e^{-2^M tx}$. Therefore,

$$\begin{aligned}\frac{t^{1-\alpha}}{\ell(t)} \int_0^{2^{-M}t^{-1}} e^{-tx}\bar{G}(x)\,dx &\leq \frac{et^{1-\alpha}}{\ell(t)} \int_0^{2^{-M}t^{-1}} e^{-2^M tx}\bar{G}(x)\,dx \\ &\leq \left(e2^{-M(1-\alpha)} \cdot \frac{\ell(2^M t)}{\ell(t)}\right) \frac{(2^M t)^{1-\alpha}}{\ell(2^M t)} \int_0^\infty e^{-2^M tx}G(x)\,dx \\ &\leq \left(\frac{\epsilon}{4\Gamma(1-\alpha)} \cdot \frac{\ell(2^M t)}{\ell(t)}\right) \frac{(2^M t)^{1-\alpha}}{\ell(2^M t)} \int_0^\infty e^{-2^M tx}G(x)\,dx. \qquad (2.13)\end{aligned}$$

By Lemma 2.2 with $2^M t$ in place of t, the portion of the right-hand side of (2.13) after the parentheses converges in probability to $\Gamma(1-\alpha)$ as $t \to \infty$. Also, because

ℓ is slowly varying, we have $\ell(2^M t)/\ell(t) \to 1$ as $t \to \infty$. Therefore, there exists T_4 such that if $t > T_4$, then

$$P\left(\frac{t^{1-\alpha}}{\ell(t)} \int_0^{2^{-M}t^{-1}} e^{-tx}\bar{G}(x)\,dx > \frac{\epsilon}{2}\right) < \frac{\epsilon}{2}. \tag{2.14}$$

It follows from (2.11), (2.12), and (2.14) that if $t > \max\{T_1, T_2, T_3, T_4\}$, then

$$P\left(\frac{t^{1-\alpha}}{\ell(t)} \int_0^{\infty} e^{-tx}\bar{G}(x)\,dx > \epsilon\right) < \epsilon,$$

which implies the lemma. □

Lemma 2.6. *We have*

$$\lim_{x\to 0} \frac{x^\alpha G(x)}{\ell(1/x)} = 1 \ \text{ in probability.}$$

Proof: Choose C_0 as in Lemma 2.4, and choose $C > \max\{C_0, 1\}$ large enough that the conclusion of Lemma 2.5 holds. For $x \geq 0$, let

$$Y(x) = \min\{G(x), C\ell(1/x)x^{-\alpha}\} - x^{-\alpha}\ell(1/x).$$

In view of Lemma 2.4, it suffices to show that

$$\lim_{x\to 0} \frac{x^\alpha Y(x)}{\ell(1/x)} = 0 \ \text{ in probability.} \tag{2.15}$$

Note that $|Y(x)| \leq Cx^{-\alpha}\ell(1/x)$ for all $x \geq 0$. By Lemmas 2.3 and 2.5,

$$\lim_{t\to\infty} \frac{t^{1-\alpha}}{\ell(t)} \int_0^{\infty} e^{-tx} Y(x)\,dx = 0 \ \text{ in probability.} \tag{2.16}$$

We proceed by contradiction. Suppose (2.15) fails to hold. Then there exists $0 < \epsilon < 1/2$ and a sequence of positive numbers $(s_n)_{n=1}^\infty$ converging to zero such that one of the following holds:

(1) We have $P(Y(s_n) > \epsilon s_n^{-\alpha}\ell(1/s_n)) > \epsilon$ for all n.
(2) We have $P(Y(s_n) < -\epsilon s_n^{-\alpha}\ell(1/s_n)) > \epsilon$ for all n.

Assume for now that we are in the first case, so $P(Y(s_n) > \epsilon s_n^{-\alpha}\ell(1/s_n)) > \epsilon$ for all n. If $Y(s_n) > \epsilon s_n^{-\alpha}\ell(1/s_n)$, then $G(s_n) > (1+\epsilon)s_n^{-\alpha}\ell(1/s_n)$. In this case, if $x < s_n$, we have

$$G(x) \geq G(s_n) > (1+\epsilon)s_n^{-\alpha}\ell(1/s_n) = (1+\epsilon)\left(\frac{x}{s_n}\right)^\alpha \frac{\ell(1/s_n)}{\ell(1/x)} \cdot x^{-\alpha}\ell(1/x).$$

Choose $\delta > 0$ small enough that $(1+\delta)^{-1}(1+\epsilon)^{1-\alpha-\delta} > 1$. If $s_n/(1+\epsilon) < x < s_n$ and if n is large enough that $1/s_n > x_0(\delta)$, then by Lemma 2.1,

$$\frac{1}{(1+\delta)(1+\epsilon)^\delta} \leq \frac{\ell(1/s_n)}{\ell(1/x)} \leq (1+\delta)(1+\epsilon)^\delta.$$

Therefore,

$$G(x) > \frac{(1+\epsilon)^{1-\alpha-\delta}}{(1+\delta)} x^{-\alpha}\ell(1/x).$$

It follows that for $s_n/(1+\epsilon) < x < s_n$, we have

$$\begin{aligned} Y(x) &> \left(\frac{(1+\epsilon)^{1-\alpha-\delta}}{(1+\delta)} - 1\right) x^{-\alpha}\ell(1/x) \\ &\geq \left(\frac{(1+\epsilon)^{1-\alpha-\delta}}{(1+\delta)} - 1\right) \frac{1}{(1+\delta)(1+\epsilon)^{\delta}} s_n^{-\alpha}\ell(1/s_n) \\ &= \eta s_n^{-\alpha}\ell(1/s_n), \end{aligned} \tag{2.17}$$

where $\eta > 0$.

Let $f : [0,\infty) \to \mathbb{R}$ be the function such that $f(x) = 0$ if either $x \leq 1/(1+\epsilon)$ or $x \geq 1$, $f((2+\epsilon)/(2+2\epsilon)) = 1$, and f is linear on the intervals $[1/(1+\epsilon), (2+\epsilon)/(2+2\epsilon)]$ and $[(2+\epsilon)/(2+2\epsilon), 1]$. Note that

$$\int_{1/(1+\epsilon)}^{1} f(x)\, dx = \frac{1}{2}\left(1 - \frac{1}{1+\epsilon}\right) = \frac{\epsilon}{2(1+\epsilon)}. \tag{2.18}$$

Let $\mathcal{A}$ be the algebra of functions of the form $\varphi(x) = a_1 e^{-t_1 x} + \cdots + a_m e^{-t_m x}$ for $x \geq 0$, where m is a nonnegative integer, $a_1, \ldots, a_m \in \mathbb{R}$, and $t_1, \ldots, t_m \geq 1$. By the Stone-Weierstrass Theorem (see, for example, Theorem D.23 on p. 346 of Cohn (1980)), the set $\mathcal{A}$ is uniformly dense in the set $C_0([0,\infty))$ of continuous functions from $[0,\infty)$ to $\mathbb{R}$ that vanish at infinity. Therefore, if we choose $\zeta = \epsilon\eta/(16\Gamma(1-\alpha)C)$, then there is a function $g \in \mathcal{A}$ such that $|g(x) - e^x f(x)| \leq \zeta$ for all $x \geq 0$. Letting $h(x) = e^{-x}g(x)$ for $x \geq 0$, we have $|h(x) - f(x)| \leq \zeta e^{-x}$ for all $x \geq 0$. Write $g(x) = a_1 e^{-t_1 x} + \cdots + a_m e^{-t_m x}$.

Choose $\theta = \min\{\epsilon/2m, 2^{1-\alpha}\epsilon\eta/8(|a_1| + \cdots + |a_m|)\} > 0$. By (2.16) we can choose n large enough that $2/s_n \geq T$, where for $t \geq T$ we have

$$P\left(\left|\frac{t^{1-\alpha}}{\ell(t)} \int_0^\infty e^{-tx} Y(x)\, dx\right| > \theta\right) < \theta.$$

It follows that with probability at least $1 - m\theta$,

$$\begin{aligned} \left|\int_0^\infty h(x/s_n) Y(x)\, dx\right| &= \left|\sum_{i=1}^m a_i \int_0^\infty e^{-(t_i+1)x/s_n} Y(x)\, dx\right| \\ &\leq \sum_{i=1}^m |a_i| \left(\frac{t_i+1}{s_n}\right)^{\alpha-1} \ell\left(\frac{t_i+1}{s_n}\right)\theta \\ &\leq \frac{\theta}{2^{1-\alpha}} s_n^{1-\alpha}\ell(1/s_n) \sum_{i=1}^m |a_i| \frac{\ell((t_i+1)/s_n)}{\ell(1/s_n)}. \end{aligned} \tag{2.19}$$

Also, using (2.6) with $1/s_n$ in place of t, we have that for sufficiently large n,

$$\begin{aligned} \left|\int_0^\infty \big(f(x/s_n) - h(x/s_n)\big) Y(x)\, dx\right| &\leq \int_0^\infty \zeta e^{-x/s_n} C x^{-\alpha}\ell(1/x)\, dx \\ &\leq 2C\Gamma(1-\alpha)\zeta s_n^{1-\alpha}\ell(1/s_n). \end{aligned} \tag{2.20}$$

Since ℓ is slowly varying, it follows from (2.19) and (2.20) that with probability at least $1 - m\theta$,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{s_n^{1-\alpha}\ell(1/s_n)} \left|\int_0^\infty f(x/s_n) Y(x)\, dx\right| \\ &\qquad \leq 2\Gamma(1-\alpha)C\zeta + \frac{\theta(|a_1| + \cdots + |a_m|)}{2^{1-\alpha}} \leq \frac{\epsilon\eta}{4}. \end{aligned} \tag{2.21}$$

However, (2.17) and (2.18) imply that for sufficiently large n, with probability at least ϵ,

$$\int_0^\infty f(x/s_n)Y(x)\,dx = \int_{s_n/(1+\epsilon)}^{s_n} f(x/s_n)Y(x)\,dx$$
$$> \eta s_n^{-\alpha}\ell(1/s_n)\int_{s_n/(1+\epsilon)}^{s_n} f(x/s_n)\,dx$$
$$= \frac{\epsilon\eta}{2(1+\epsilon)}s_n^{1-\alpha}\ell(1/s_n),$$

which contradicts (2.21) because $m\theta < \epsilon$.

It remains to consider the second case. Assume $P(Y(s_n) < -\epsilon s_n^{-\alpha}\ell(1/s_n)) > \epsilon$ for all n. If $Y(s_n) < -\epsilon s_n^{-\alpha}\ell(1/s_n)$, then $G(s_n) < (1-\epsilon)s_n^{-\alpha}\ell(1/s_n)$. In this case, if $x > s_n$, then

$$G(x) \le G(s_n) < (1-\epsilon)s_n^{-\alpha}\ell(1/s_n) = (1-\epsilon)\left(\frac{x}{s_n}\right)^\alpha \frac{\ell(1/s_n)}{\ell(1/x)}\cdot x^{-\alpha}\ell(1/x).$$

Choose $\delta > 0$ small enough that $(1+\delta)(1-\epsilon)^{1-\alpha-\delta} < 1$. If $s_n < x < s_n/(1-\epsilon)$ and if n is large enough that $(1-\epsilon)/s_n > x_0(\delta)$, then by Lemma 2.1,

$$\frac{(1-\epsilon)^\delta}{1+\delta} \le \frac{\ell(1/s_n)}{\ell(1/x)} \le \frac{1+\delta}{(1-\epsilon)^\delta}.$$

Therefore,

$$G(x) < (1+\delta)(1-\epsilon)^{1-\alpha-\delta}x^{-\alpha}\ell(1/x).$$

It follows that for $s_n < x < s_n/(1-\epsilon)$, we have

$$\begin{aligned} Y(x) &< \big((1+\delta)(1-\epsilon)^{1-\alpha-\delta}-1\big)x^{-\alpha}\ell(1/x) \\ &\le \big((1+\delta)(1-\epsilon)^{1-\alpha-\delta}-1\big)(1+\delta)^{-1}(1-\epsilon)^{\alpha+\delta}s_n^{-\alpha}\ell(1/s_n) \\ &= -\eta s_n^{-\alpha}\ell(1/s_n), \end{aligned} \tag{2.22}$$

where $\eta > 0$.

This time, let $f : [0,\infty) \to \mathbb{R}$ be the function such that $f(x) = 0$ if $x \le 1$ or $x \ge 1/(1-\epsilon)$, $f((2-\epsilon)/(2-2\epsilon)) = 1$, and f is linear on $[1,(2-\epsilon)/(2-2\epsilon)]$ and $[(2-\epsilon)/(2-2\epsilon), 1/(1-\epsilon)]$. We have

$$\int_1^{1/(1-\epsilon)} f(x)\,dx = \frac{1}{2}\left(\frac{1}{1-\epsilon}-1\right) = \frac{\epsilon}{2(1-\epsilon)}. \tag{2.23}$$

Define g and θ as in the previous case. Then (2.19), (2.20), and (2.21) hold as before. However, (2.22) and (2.23) imply that for sufficiently large n, with probability at least ϵ,

$$\begin{aligned} \int_0^\infty f(x/s_n)Y(x)\,dx &= \int_{s_n}^{s_n/(1-\epsilon)} f(x/s_n)Y(x)\,dx \\ &< -\eta s_n^{-\alpha}\ell(1/s_n)\int_{s_n}^{s_n/(1-\epsilon)} f(x/s_n)\,dx \\ &= -\frac{\epsilon\eta}{2(1-\epsilon)}s_n^{1-\alpha}\ell(s_n) \end{aligned} \tag{2.24}$$

which again contradicts (2.21) because $m\theta < \epsilon$. $\square$

Proof of Theorem 1.2: Fix $r \in \mathbb{N}$. It follows from (1.4) and (2.4) that given $\epsilon > 0$, for sufficiently large n we have

$$P\left(\left|\sum_{s=r}^{\infty} K_{n,s} - \sum_{s=r}^{\infty} \Psi_s(n)\right| < \frac{\epsilon}{2} n^{\alpha} \ell(n)\right) > 1 - \frac{\epsilon}{2} \tag{2.25}$$

and

$$P\left(\left|\sum_{s=r+1}^{\infty} K_{n,s} - \sum_{s=r+1}^{\infty} \Psi_s(n)\right| < \frac{\epsilon}{2} n^{\alpha} \ell(n)\right) > 1 - \frac{\epsilon}{2} \tag{2.26}$$

Subtracting (2.26) from (2.25) gives that

$$P\left(\left|\frac{K_{n,r}}{n^{\alpha}\ell(n)} - \frac{\Phi_r(n)}{n^{\alpha}\ell(n)}\right| < \epsilon\right) > 1 - \epsilon$$

for sufficiently large n, Therefore, it suffices to show that

$$\lim_{t \to \infty} \frac{\Phi_r(t)}{t^{\alpha}\ell(t)} = \frac{\alpha\Gamma(r - \alpha)}{r!} \quad \text{in probability.} \tag{2.27}$$

Let $\theta > 0$ be arbitrary. Because $\sum_{r=1}^{\infty} \alpha\Gamma(r-\alpha)/r! = \Gamma(1-\alpha)$, we can choose N large enough that

$$\sum_{r=1}^{N} \frac{\alpha\Gamma(r-\alpha)}{r!} > \Gamma(1-\alpha) - \frac{\theta}{2}. \tag{2.28}$$

Let $\eta = \min\{\theta/(N+1), \theta/(4\Gamma(1-\alpha))\}$. Note that we can choose a sufficiently large integer L, then a sufficiently small positive number δ (much smaller than $1/L$), then a sufficiently large integer M (much larger than $1/\delta$), then a sufficiently small positive number ϵ (much smaller than $1/M$) such that

$$\left(\frac{L}{L+2}\right)^r \left((1-\epsilon)^2 - \frac{4\epsilon M}{\alpha}\right) \int_{\delta(L+1)}^{\delta(M+1)} e^{-y} y^{r-\alpha-1}\, dy > (1-\eta)\Gamma(r-\alpha) \tag{2.29}$$

for $1 \leq r \leq N$. By Lemma 2.6, we can choose $T_1 > 0$ sufficiently large that if $t \geq T_1$, then

$$P\big((1-\epsilon)x^{-\alpha}\ell(1/x) \leq G(x) \leq (1+\epsilon)x^{-\alpha}\ell(1/x) \\ \text{for } x = L\delta/t, (L+1)\delta/t, \ldots, M\delta/t\big) > 1 - \eta.$$

By Lemma 2.1, we can choose $T_2 > 0$ sufficiently large that if $t \geq T_2$ and $L\delta/t \leq x \leq M\delta/t$, then

$$1 - \epsilon \leq \frac{\ell(1/x)}{\ell(t)} \leq 1 + \epsilon.$$

If $t \geq \max\{T_1, T_2\}$, then with probability at least $1-\eta$, we have, using (2.2),

$$\begin{aligned}
\frac{\Phi_r(t)}{t^\alpha \ell(t)} &= \frac{t^{r-\alpha}}{r!\ell(t)} \sum_{j=1}^{\infty} P_j^r e^{-tP_j} \\
&\geq \frac{t^{r-\alpha}}{r!\ell(t)} \sum_{k=L}^{M-1} \left(\frac{k\delta}{t}\right)^r e^{-(k+1)\delta} \big(G(k\delta/t) - G((k+1)\delta/t)\big) \\
&\geq \frac{t^{-\alpha}\delta^r}{r!\ell(t)} \sum_{k=L}^{M-1} k^r e^{-(k+1)\delta} \Bigg((1-\epsilon)\left(\frac{k\delta}{t}\right)^{-\alpha} \ell\left(\frac{t}{k\delta}\right) \\
&\qquad\qquad - (1+\epsilon)\left(\frac{(k+1)\delta}{t}\right)^{-\alpha} \ell\left(\frac{t}{(k+1)\delta}\right)\Bigg) \\
&\geq \frac{\delta^{r-\alpha}}{r!} \sum_{k=L}^{M-1} k^r e^{-(k+1)\delta} \left(\frac{(1-\epsilon)^2}{k^\alpha} - \frac{(1+\epsilon)^2}{(k+1)^\alpha}\right).
\end{aligned} \tag{2.30}$$

For $k \leq M-1$,

$$\begin{aligned}
\frac{(1-\epsilon)^2}{k^\alpha} - \frac{(1+\epsilon)^2}{(k+1)^\alpha} &= (1-\epsilon)^2\left(\frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha}\right) + \frac{1}{(k+1)^\alpha}\big((1-\epsilon)^2 - (1+\epsilon)^2\big) \\
&\geq \frac{\alpha(1-\epsilon)^2}{(k+1)^{\alpha+1}} - \frac{4\epsilon}{(k+1)^\alpha} \\
&\geq \frac{\alpha}{(k+1)^{\alpha+1}}\left((1-\epsilon)^2 - \frac{4\epsilon M}{\alpha}\right).
\end{aligned}$$

Therefore, if $1 \leq r \leq N$ and $t \geq \max\{T_1, T_2\}$, then with probability at least $1-\eta$,

$$\frac{\Phi_r(t)}{t^\alpha \ell(t)} \geq \left((1-\epsilon)^2 - \frac{4\epsilon M}{\alpha}\right) \frac{\alpha \delta^{r-\alpha}}{r!} \sum_{k=L}^{M-1} \frac{k^r}{(k+1)^{\alpha+1}} e^{-(k+1)\delta}.$$

If $r \geq 2$ and $k \geq L$ then

$$\begin{aligned}
\frac{k^r}{(k+1)^{\alpha+1}} e^{-(k+1)\delta} &\geq \left(\frac{L}{L+2}\right)^r (k+2)^{r-\alpha-1} e^{-(k+1)\delta} \\
&\geq \left(\frac{L}{L+2}\right)^r \int_{k+1}^{k+2} x^{r-\alpha-1} e^{-\delta x}\, dx,
\end{aligned} \tag{2.31}$$

and if $r = 1$ and $k \geq L$ then

$$\begin{aligned}
\frac{k^r}{(k+1)^{\alpha+1}} e^{-(k+1)\delta} &\geq \left(\frac{L}{L+1}\right)^r (k+1)^{r-\alpha-1} e^{-(k+1)\delta} \\
&\geq \left(\frac{L}{L+2}\right)^r \int_{k+1}^{k+2} x^{r-\alpha-1} e^{-\delta x}\, dx.
\end{aligned}$$

Thus, if $1 \leq r \leq N$ and $t \geq \max\{T_1, T_2\}$, then with probability at least $1 - \eta$, we have

$$\begin{aligned}
\frac{\Phi_r(t)}{t^\alpha \ell(t)} &\geq \left(\frac{L}{L+2}\right)^r \left((1-\epsilon)^2 - \frac{4\epsilon M}{\alpha}\right) \frac{\alpha \delta^{r-\alpha}}{r!} \int_{L+1}^{M+1} x^{r-\alpha-1} e^{-\delta x} \, dx \\
&= \left(\frac{L}{L+2}\right)^r \left((1-\epsilon)^2 - \frac{4\epsilon M}{\alpha}\right) \frac{\alpha}{r!} \int_{\delta(L+1)}^{\delta(M+1)} e^{-y} y^{r-\alpha-1} \, dy. \\
&> \frac{(1-\eta)\alpha\Gamma(r-\alpha)}{r!},
\end{aligned} \tag{2.32}$$

where the last inequality uses (2.29).

Since $t \mapsto \Phi(t)$ is nondecreasing and ℓ is slowly varying, (1.4) and (2.3) imply that $\Phi(t)/(t^\alpha \ell(t))$ converges in probability to $\Gamma(1-\alpha)$ as $t \to \infty$. Therefore, there exists T_3 such that if $t \geq T_3$, then

$$P\left(\frac{\Phi(t)}{t^\alpha \ell(t)} \leq (1+\eta)\Gamma(1-\alpha)\right) > 1 - \eta. \tag{2.33}$$

Therefore, combining (2.32) and (2.33), if $1 \leq r \leq N$ and $t \geq \max\{T_1, T_2, T_3\}$, then with probability at least $1 - (N+1)\eta \geq 1 - \theta$,

$$\begin{aligned}
\frac{\Phi_r(t)}{t^\alpha \ell(t)} &\leq \frac{1}{t^\alpha \ell(t)} \left(\Phi(t) - \sum_{\substack{s=1 \\ s \neq r}}^N \Phi_s(t)\right) \\
&\leq (1+\eta)\Gamma(1-\alpha) - (1-\eta) \sum_{\substack{s=1 \\ s \neq r}}^N \frac{\alpha\Gamma(s-\alpha)}{s!} \\
&= \Gamma(1-\alpha) - \sum_{\substack{s=1 \\ s \neq r}}^N \frac{\alpha\Gamma(s-\alpha)}{s!} + \eta\left(\Gamma(1-\alpha) + \sum_{\substack{s=1 \\ s \neq r}}^N \frac{\alpha\Gamma(s-\alpha)}{s!}\right) \\
&\leq \frac{\alpha\Gamma(r-\alpha)}{r!} + \frac{\theta}{2} + 2\eta\Gamma(1-\alpha) \\
&\leq \frac{\alpha\Gamma(r-\alpha)}{r!} + \theta.
\end{aligned} \tag{2.34}$$

using (2.28). The result (2.27) for $r = 1, \ldots, N$ now follows from (2.32) and (2.34). Since N can be chosen arbitrarily large, the result holds for all positive integers r. $\square$

3. Description of Example 1.3

We specify a random sequence $P_1 \geq P_2 \geq \ldots$ such that $\sum_{j=1}^\infty P_j = 1$ a.s. in the following way:

(1) Begin with any deterministic sequence $q_1 \geq q_2 \geq \ldots$ such that $\sum_{j=1}^\infty q_j < 1/2$ and such that if $g(x) = \max\{j : q_j \geq x\}$, then $\lim_{x \to 0} x^\alpha g(x) = 1$.

(2) Given a positive integer n_1, we can define, for all integers $k \geq 2$, the integer $n_k = 2^{2^{2^{n_{k-1}}}}$. Choose n_1 large enough that $\sum_{k=1}^\infty n_k^{\alpha-1} < 1/2$.

(3) Define a sequence of independent random variables $(R_k)_{k=1}^\infty$ such that R_k has the uniform distribution on $\{1, 2, \ldots, n_k\}$ for all k. Then for all $k \in \mathbb{N}$, add the number $1/(n_k 2^{2^{R_k}})$ to the sequence $\lfloor 2^{2^{R_k}} n_k^\alpha \rfloor$ times.

(4) Add the number

$$1 - \sum_{j=1}^{\infty} q_j - \sum_{k=1}^{\infty} \frac{1}{n_k 2^{2^{R_k}}} \lfloor 2^{2^{R_k}} n_k^\alpha \rfloor$$

to the sequence to make the numbers sum to one.

(5) Order the numbers and relabel them $P_1 \geq P_2 \geq \dots$.

Using the method described in the introduction, define an exchangeable random partition Π whose asymptotic block frequencies are almost surely given by this sequence $(P_j)_{j=1}^{\infty}$. The next two lemmas show that Π satisfies the conditions of Example 1.3.

Lemma 3.1. *For the sequence* $(P_j)_{j=1}^{\infty}$ *defined above, if* $G(x) = \max\{j : P_j \geq x\}$, *then*

$$\lim_{x \to 0} x^\alpha G(x) = 1 \quad \text{in probability.}$$

Proof: Let $G'(x)$ denote the number of terms that were added to the sequence in step 3 of the above construction that are greater than or equal to x. Because $\lim_{x\to 0} x^\alpha g(x) = 1$ by step 1 of the construction, it suffices to show that

$$\lim_{x \to 0} x^\alpha G'(x) = 0 \quad \text{in probability.} \tag{3.1}$$

Let $\epsilon > 0$. Suppose $1/n_{k+1} \leq x \leq 1/n_k$. Because $R_j \leq n_j$, there can be at most $2^{2^{n_j}} n_j^\alpha$ terms in the sequence that equal $1/(n_j 2^{2^{R_j}})$ for $j = 1, \dots, k-1$. Therefore,

$$G'(x) \leq \sum_{j=1}^{k-1} 2^{2^{n_j}} n_j^\alpha + 2^{2^{R_k}} n_k^\alpha \mathbf{1}_{\{1/(n_k 2^{2^{R_k}}) \geq x\}}. \tag{3.2}$$

By the choice of n_k, we have

$$\sum_{j=1}^{k-1} 2^{2^{n_j}} n_j^\alpha \leq \frac{\epsilon}{2} n_k^\alpha \leq \frac{\epsilon}{2} x^{-\alpha}$$

for sufficiently large k. The second term on the right-hand side of (3.2) will be at most $(\epsilon/2)x^{-\alpha}$ unless we have both $1/(n_k 2^{2^{R_k}}) \geq x$ and $2^{2^{R_k}} n_k^\alpha \geq (\epsilon/2)x^{-\alpha}$ or, equivalently, unless

$$\log_2 \log_2 \left(\frac{\epsilon}{2x^\alpha n_k^\alpha} \right) \leq R_k \leq \log_2 \log_2 \left(\frac{1}{x n_k} \right).$$

Because R_k has a uniform distribution on $\{1, \dots, n_k\}$, the probability that R_k falls in this interval is at most

$$\frac{1}{n_k} \left(1 + \log_2 \log_2 \left(\frac{1}{x n_k} \right) - \log_2 \log_2 \left(\frac{\epsilon}{2x^\alpha n_k^\alpha} \right) \right). \tag{3.3}$$

Note that for all real numbers $z > 1$, we have

$$\log_2 \log_2 z - \log_2 \log_2 z^\alpha = \log_2 \left(\frac{\log_2 z}{\log_2 z^\alpha} \right) = \log_2 \left(\frac{1}{\alpha} \right).$$

By applying this result when $z = 1/(x n_k)$, we see that the probability in (3.3) tends to zero as $k \to \infty$. It follows that $\lim_{x \to \infty} P(G'(x) > \epsilon x^{-\alpha}) = 0$ for all $\epsilon > 0$, and (3.1) follows. □

Lemma 3.2. *For the random partition* Π *defined above, if* Π_n *denotes the restriction of* Π *to* $\{1, \dots, n\}$ *and* K_n *denotes the number of blocks of* Π_n*, then there exists a constant* $C > 0$ *such that*

$$\lim_{k \to \infty} P\big(n_k^{-\alpha} K_{n_k} \geq \Gamma(1-\alpha) + C\big) = 1.$$

Proof: We use Poissonization. Let $(N(t), t \geq 0)$ be a rate one Poisson process, and let $\Phi(t) = E[K_{N(t)} | (P_j)_{j=1}^{\infty}]$. By (2.3), it suffices to show that there is a $C > 0$ such that

$$\liminf_{k \to \infty} n_k^{-\alpha} \Phi(n_k) \geq \Gamma(1-\alpha) + C \quad \text{a.s.} \tag{3.4}$$

For all $k \in \mathbb{N}$, designate $\lfloor 2^{2^{R_k}} n_k^{\alpha} \rfloor$ blocks of Π with asymptotic frequency $1/(n_k 2^{2^{R_k}})$ as marked blocks, while the other blocks of Π will be unmarked. If there are more than $\lfloor 2^{2^{R_k}} n_k^{\alpha} \rfloor$ blocks with asymptotic frequency $1/(n_k 2^{2^{R_k}})$ because $q_j = 1/(n_k 2^{2^{R_k}})$ for some j, then choose at random the blocks to mark. Note that the marked blocks correspond to the terms P_k that were added in step 3 of the above construction. The unmarked blocks all have asymptotic frequency q_j for some j, except for the block added in step 4 of the construction. Let $\Phi'(t)$ be the expected number of marked blocks of $\Pi_{N(t)}$ conditional on $(P_j)_{j=1}^{\infty}$, and let $\Phi''(t)$ be the expected number of unmarked blocks of $\Pi_{N(t)}$ conditional on $(P_j)_{j=1}^{\infty}$. Note that $\Phi(t) = \Phi'(t) + \Phi''(t)$. By Proposition 1.1 and (2.3), we have

$$\lim_{k \to \infty} n_k^{-\alpha} \Phi''(n_k) = \Gamma(1-\alpha) \quad \text{a.s.} \tag{3.5}$$

The number of integers in the set $\{1, \dots, N(n_k)\}$ that are in a block of Π with asymptotic frequency $1/(n_k 2^{2^r})$ has a Poisson distribution with mean 2^{-2^r}. Therefore, on the event $\{R_k = r\}$, we have

$$\Phi'(n_k) \geq \lfloor 2^{2^r} n_k^{\alpha} \rfloor (1 - e^{-2^{-2^r}}).$$

Since $x^{-1}(1 - e^{-x})$ is bounded away from zero for all $x \leq 1/4$, it follows that there is a constant $C > 0$ such that $n_k^{-\alpha} \Phi'(n_k) \geq C$ a.s. for all k. This fact, combined with (3.5), implies (3.4). $\square$

4. Description of Example 1.5

We begin by specifying a deterministic sequence of numbers $p_1 \geq p_2 \geq \dots$ such that $\sum_{j=1}^{\infty} p_j = 1$ as follows:

(1) Begin with any sequence $q_1 \geq q_2 \geq \dots$ such that if $g(x) = \max\{j : q_j \geq x\}$, then

$$\lim_{x \to 0} x (\log x)^2 g(x) = 1. \tag{4.1}$$

It is not difficult to see that such sequences exist. One arises, for example, in Basdevant and Goldschmidt (2008).

(2) Choose any integer j such that

$$\sum_{k=j+1}^{\infty} q_k < 1 - \sum_{n=2}^{\infty} n^{-9/2}.$$

Then remove the terms $q_1, \dots q_j$ from the sequence.

(3) For all $n \geq 2$, add the number e^{-n^3} to the list $\lfloor n^{-9/2} e^{n^3} \rfloor$ times.

(4) Add the number

$$1 - \sum_{k=j+1}^{\infty} q_k - \sum_{n=2}^{\infty} e^{-n^3} \lfloor n^{-9/2} e^{n^3} \rfloor$$

to the sequence to make the numbers in the new sequence sum to one.

(5) Order the numbers and relabel them $p_1 \geq p_2 \geq \ldots$.

Using the method described in the introduction, define an exchangeable random partition Π whose asymptotic block frequencies are almost surely given by this sequence $(p_j)_{j=1}^{\infty}$. The next two lemmas establish that Π satisfies the conditions of Example 1.5.

Lemma 4.1. *For the random partition Π defined above, if Π_n denotes the restriction of Π to $\{1, \ldots, n\}$ and K_n denotes the number of blocks of Π_n, then*

$$\lim_{n\to\infty} \frac{(\log n)K_n}{n} = 1 \quad \text{a.s.}$$

Proof: We again use Poissonization. Let $(N(t), t \geq 0)$ be a rate one Poisson process, and let $\Phi(t) = E[K_{N(t)}]$. By (2.3), it suffices to show that

$$\lim_{t\to\infty} \frac{(\log t)\Phi(t)}{t} = 1. \tag{4.2}$$

For all $n \geq 2$, designate $\lfloor n^{-9/2} e^{n^3} \rfloor$ blocks of Π with asymptotic frequency e^{-n^3} as marked blocks, while the others are unmarked blocks. If there are more than $\lfloor n^{-9/2} e^{n^3} \rfloor$ blocks with asymptotic frequency e^{-n^3} because $q_k = e^{-n^3}$ for some k, then choose at random the blocks to mark. Note that the marked blocks correspond to the terms p_k that were added in step 3 of the construction above. The unmarked blocks all have asymptotic frequency q_k for some $k > j$, except for the one unmarked block that is added in step 4 of the construction. Let $\Phi'(t)$ be the expected number of marked blocks of $\Pi_{N(t)}$, and let $\Phi''(t)$ be the expected number of unmarked blocks of $\Pi_{N(t)}$. Note that $\Phi(t) = \Phi'(t) + \Phi''(t)$. In view of (4.1), we can apply Proposition 1.4 with $\ell(t) = (\log t)^{-2}$ for $t > 1$ in combination with (2.3) to get

$$\lim_{t\to\infty} \frac{(\log t)\Phi''(t)}{t} = 1. \tag{4.3}$$

That $q_1, \ldots, q_j$ were deleted and one unmarked block was added does not affect this conclusion.

Now, choose t such that $e^{(n-1)^3} < t \leq e^{n^3}$. The number of marked blocks of Π with asymptotic frequency at least $e^{-(n-1)^3}$ is

$$\sum_{k=1}^{n-1} \lfloor k^{-9/2} e^{k^3} \rfloor \leq C_1 n^{-9/2} e^{(n-1)^3} \leq C_1 n^{-9/2} t,$$

where C_1 is a positive constant that does not depend on n. This bound holds because the sum is dominated by the largest term. If a block of Π has asymptotic frequency q, then the probability that at least one of the first $N(t)$ integers is in the block is $1 - e^{-qt} \leq qt$. Therefore, the expected number of marked blocks of $\Pi_{N(t)}$ with asymptotic frequency e^{-n^3} or smaller is at most

$$\sum_{k=n}^{\infty} (e^{-k^3} t) \cdot k^{-9/2} e^{k^3} = t \sum_{k=n}^{\infty} k^{-9/2} \leq C_2 n^{-7/2} t,$$

where C_2 is another positive constant that does not depend on n. Therefore, $\Phi'(t) \leq C_1 n^{-9/2} t + C_2 n^{-7/2} t$. Since $\log t \leq n^3$, it follows that

$$\lim_{t \to \infty} \frac{(\log t)\Phi'(t)}{t} = 0. \tag{4.4}$$

Now (4.2) follows from (4.4) and (4.3). □

Lemma 4.2. *For the random partition Π defined above, if Π_n denotes the restriction of Π to $\{1, \dots, n\}$ and $K_{n,r}$ denotes the number of blocks of Π_n of size r, then for $r \geq 2$, the quantity $n^{-1}(\log n)^2 K_{n,r}$ does not converge to $1/[r(r-1)]$ in probability as $n \to \infty$.*

Proof: We consider the sequence $(K_{\lfloor m_n \rfloor, r})_{n=1}^{\infty}$, where $m_n = e^{n^3}$ for all n. Let $(N(t), t \geq 0)$ be a rate one Poisson process. There are at least $\lfloor n^{-9/2} e^{n^3} \rfloor$ blocks of Π with asymptotic frequency e^{-n^3}. Order these blocks at random, and then let $A_{i,n}$ be the event that the ith of these blocks contains exactly r of the integers $1, \dots, N(m_n)$. Because the number of the integers $\{1, \dots, N(n_m)\}$ in one of these blocks has a Poisson distribution with mean 1, we have $P(A_{i,n}) = e^{-1}/r!$ for all i and n. Also, for any n, the events $A_{i,n}$ for $1 \leq i \leq \lfloor n^{-9/2} e^{n^3} \rfloor$ are independent. It follows that for all n, the random variable $K_{N(m_n),r}$ stochastically dominates a Binomial$(\lfloor n^{-9/2} e^{n^3} \rfloor, e^{-1}/r!)$ random variable. It now follows from standard large deviations estimates that

$$\lim_{n \to \infty} P\left(K_{N(m_n),r} \geq \frac{2e^{-1}}{3r!} n^{-9/2} e^{n^3} \right) = 1. \tag{4.5}$$

Because $N(m_n)$ has the Poisson distribution with mean m_n, we have $\mathrm{Var}(N(m_n)) = m_n$ and therefore $E[|N(m_n) - m_n|] \leq m_n^{1/2}$. Since

$$|K_{N(m_n),r} - K_{\lfloor m_n \rfloor, r}| \leq |N(m_n) - \lfloor m_n \rfloor|,$$

it follows that $E[|K_{N(m_n),r} - K_{\lfloor m_n \rfloor,r}|] \leq e^{n^3/2} + 1$. Combining this result with Markov's inequality gives

$$\lim_{n \to \infty} P\left(|K_{N(m_n),r} - K_{\lfloor m_n \rfloor,r}| > \frac{e^{-1}}{3r!} n^{-9/2} e^{n^3} \right) = 0. \tag{4.6}$$

Combining (4.5) and (4.6) gives

$$\lim_{n \to \infty} P\left(K_{\lfloor m_n \rfloor,r} \geq \frac{e^{-1}}{3r!} n^{-9/2} e^{n^3} \right) = 1.$$

Since $m_n (\log m_n)^{-2} = n^{-6} e^{n^3}$, the result follows. □

5. Proof of Theorem 1.6

We will assume that $(\Psi_n(t), t \geq 0)$ is obtained from Kingman's coalescent $(\Theta_n(t), t \geq 0)$ as in (1.12). For any partition π of $\{1, \dots, n\}$, let $|\pi|$ denote the number of blocks of π. For $1 \leq k \leq n$, let $T_k = \inf\{t : |\Theta_n(t)| = k\}$.

Lemma 5.1. *For all $\epsilon > 0$, there exists a positive constant C such that with probability at least $1 - \epsilon$, we have*

$$\left| T_k - \left(\frac{2}{k} - \frac{2}{n} \right) \right| \leq \frac{C}{n^{9/8}}$$

for all integers k such that $n^{3/4} \leq k \leq n$.

Proof: If $2 \leq k \leq n$, then $T_{k-1} - T_k$ has an exponential distribution with rate $\binom{k}{2}$. Since $T_n = 0$, it follows that

$$E[T_k] = \sum_{j=k+1}^{n} E[T_{j-1} - T_j] = \sum_{j=k+1}^{n} \frac{2}{j(j-1)} = \sum_{j=k+1}^{n} \left(\frac{2}{j-1} - \frac{2}{j} \right) = \frac{2}{k} - \frac{2}{n}.$$

For $1 \leq k \leq n$, let $Y_k = T_k - E[T_k]$. Note that $Y_{k-1} - Y_k = T_{k-1} - T_k - 2/[k(k-1)]$, and these increments are independent. Therefore,

$$\mathrm{Var}(Y_k) = \sum_{j=k+1}^{n} \mathrm{Var}(Y_{j-1} - Y_j) = \sum_{j=k+1}^{n} \mathrm{Var}(T_{j-1} - T_j) = \sum_{j=k+1}^{n} \frac{4}{j^2(j-1)^2} \leq \frac{C_1}{k^3}$$

for some positive constant C_1. By Kolmogorov's Maximal Inequality,

$$P\left(\max_{k \leq j \leq n} |Y_j| > \frac{C}{k^{3/2}} \right) \leq \frac{k^3}{C^2} \cdot \frac{C_1}{k^3} = \frac{C_1}{C^2},$$

which is less than ϵ if we take C sufficiently large. The result follows by taking $k = \lceil n^{3/4} \rceil$, in which case $C/k^{3/2} \leq C/n^{9/8}$. $\square$

For $1 \leq k \leq n$, let $U_k = \inf\{t : |\Psi_n(t)| = k\}$. Define the function $g : [0, \infty) \to [0, \infty)$ by $g(t) = (1-\alpha)^{-(1-\alpha)} t^{1-\alpha}$, where $\alpha = \gamma/(1+\gamma)$. It follows from (1.12) that for all $t \geq 0$,

$$\Psi_n(g(t)) = \Theta_n\left(\frac{g(t)^{\gamma+1}}{\gamma+1} \right) = \Theta_n\left(\frac{t^{(1-\alpha)(\gamma+1)}}{(1-\alpha)^{(1-\alpha)(\gamma+1)}(\gamma+1)} \right) = \Theta_n(t).$$

Therefore, $U_k = g(T_k)$ for all k.

Let

$$L_n = \sum_{k=2}^{n} k(U_{k-1} - U_k).$$

Note that L_n is the sum of the lengths of all branches in the coalescent tree because $U_{k-1} - U_k$ is the amount of time for which there are exactly k lineages. Let $m = \lceil n^{3/4} \rceil + 1$, and let

$$L_n' = \sum_{k=m}^{n} k(U_{k-1} - U_k),$$

which is the total length of all branches in the coalescent tree when the tree is truncated at the point where the number of lineages reaches $\lceil n^{3/4} \rceil$.

Lemma 5.2. *We have*

$$\lim_{n \to \infty} \frac{L_n'}{n^\alpha} = \frac{2^{1-\alpha}(1-\alpha)^\alpha \pi}{\sin(\pi\alpha)} \quad \text{in probability.}$$

Proof: Let $\epsilon > 0$. By Lemma 5.1, there is a constant C such that with probability $1 - \epsilon$, we have

$$\frac{2}{k} - \frac{2}{n} - \frac{C}{n^{9/8}} \leq T_k \leq \frac{2}{k} - \frac{2}{n} + \frac{C}{n^{9/8}}$$

whenever $n^{3/4} \leq k \leq n$. Note that $U_n = 0$ and L'_n is an increasing function of U_k for $m-1 \leq k \leq n-1$. Therefore, with probability at least $1-\epsilon$,

$$\begin{aligned} L'_n &= \sum_{k=m}^{n} k(g(T_{k-1}) - g(T_k)) \\ &\leq \sum_{k=m}^{n} k\bigg(g\bigg(\frac{2}{k-1} - \frac{2}{n} + \frac{C}{n^{9/8}}\bigg) - g\bigg(\frac{2}{k} - \frac{2}{n} + \frac{C}{n^{9/8}}\bigg)\bigg) \\ &\leq ng\bigg(\frac{2}{n-1} - \frac{2}{n} + \frac{C}{n^{9/8}}\bigg) + \sum_{k=m}^{n-1} k\bigg(\frac{2}{k-1} - \frac{2}{k}\bigg)g'\bigg(\frac{2}{k} - \frac{2}{n} + \frac{C}{n^{9/8}}\bigg), \end{aligned} \tag{5.1}$$

where the last equality uses that $g'(t)$ is a decreasing function of t because $0 < \alpha < 1$. The first term on the right-hand side of (5.1) is $O(n^{1-9(1-\alpha)/8})$ and therefore is $o(n^\alpha)$. Since $g'(t) = (1-\alpha)^\alpha t^{-\alpha}$, the second term on the right-hand side of (5.1) is equal to

$$\sum_{k=m}^{n-1} \frac{2(1-\alpha)^\alpha}{k-1}\bigg(\frac{2}{k} - \frac{2}{n} + \frac{C}{n^{9/8}}\bigg)^{-\alpha} \leq 2^{1-\alpha}(1-\alpha)^\alpha \sum_{k=m}^{n-1} \frac{1}{k-1}\bigg(\frac{1}{k} - \frac{1}{n}\bigg)^{-\alpha}. \tag{5.2}$$

For all k such that $m \leq k \leq n-1$,

$$\frac{1}{k-1}\bigg(\frac{1}{k} - \frac{1}{n}\bigg)^{-\alpha} = \bigg(\frac{k+1}{k-1}\bigg)\frac{1}{k+1}\bigg(\frac{1}{k} - \frac{1}{n}\bigg)^{-\alpha} \leq \frac{m+1}{m-1}\int_k^{k+1} \frac{1}{x}\bigg(\frac{1}{x} - \frac{1}{n}\bigg)^{-\alpha} dx.$$

Therefore, the second term on the right-hand side of (5.1) is at most

$$2^{1-\alpha}(1-\alpha)^\alpha\bigg(\frac{m-1}{m+1}\bigg)\int_0^n \frac{1}{x}\bigg(\frac{1}{x} - \frac{1}{n}\bigg)^{-\alpha} dx.$$

By making the substitution $y = x/n$, we get

$$\begin{aligned} \int_0^n \frac{1}{x}\bigg(\frac{1}{x} - \frac{1}{n}\bigg)^{-\alpha} dx = n^\alpha \int_0^1 \frac{1}{y}\bigg(\frac{1}{y} - 1\bigg)^{-\alpha} dy = n^\alpha \int_0^1 y^{\alpha-1}(1-y)^{-\alpha}\, dy \\ = n^\alpha\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi n^\alpha}{\sin(\pi\alpha)}, \end{aligned} \tag{5.3}$$

where the last step uses Euler's Reflection Formula (see, for example, p. 9 of Andrews et al. (1999)). Therefore, there exists a sequence $(a_n)_{n=1}^\infty$ tending to zero such that with probability at least $1-\epsilon$,

$$\frac{L'_n}{n^\alpha} \leq \frac{2^{1-\alpha}(1-\alpha)^\alpha\pi}{\sin(\pi\alpha)} + a_n. \tag{5.4}$$

Likewise, for the lower bound, let $M = \max\{k : 2/k - 2/n - C/n^{9/8} > 0\}$. Then with probability at least $1-\epsilon$, we have

$$\begin{aligned}
L'_n &= \sum_{k=m}^{n} k(g(T_{k-1}) - g(T_k)) \\
&\geq \sum_{k=m}^{M} k\left(g\left(\frac{2}{k-1} - \frac{2}{n} - \frac{C}{n^{9/8}}\right) - g\left(\frac{2}{k} - \frac{2}{n} - \frac{C}{n^{9/8}}\right)\right) \\
&\geq \sum_{k=m}^{M} k\left(\frac{2}{k-1} - \frac{2}{k}\right) g'\left(\frac{2}{k-1} - \frac{2}{n} - \frac{C}{n^{9/8}}\right) \\
&\geq \sum_{k=m}^{M} \frac{2}{k-1}(1-\alpha)^{\alpha}\left(\frac{2}{k-1} - \frac{2}{n}\right)^{-\alpha} \\
&= 2^{1-\alpha}(1-\alpha)^{\alpha} \sum_{m-1}^{M-1} \frac{1}{k}\left(\frac{1}{k} - \frac{1}{n}\right)^{-\alpha}.
\end{aligned}$$

For all k such that $m-1 \leq k \leq M-1$,

$$\frac{1}{k}\left(\frac{1}{k} - \frac{1}{n}\right)^{-\alpha} = \left(\frac{k-1}{k}\right)\frac{1}{k-1}\left(\frac{1}{k} - \frac{1}{n}\right)^{-\alpha} \geq \frac{m-2}{m-1}\int_{k-1}^{k} \frac{1}{x}\left(\frac{1}{x} - \frac{1}{n}\right)^{-\alpha} dx.$$

Since $m/n \to 0$ and $M/n \to 1$ as $n \to \infty$, it now follows from (5.3) that there is a sequence $(b_n)_{n=1}^{\infty}$ tending to zero such that with probability at least $1-\epsilon$,

$$\frac{L'_n}{n^{\alpha}} \geq \frac{2^{1-\alpha}(1-\alpha)^{\alpha}\pi}{\sin(\pi\alpha)} - b_n. \tag{5.5}$$

The result now follows from (5.4) and (5.5). □

Lemma 5.3. *We have*

$$\lim_{n\to\infty} \frac{L_n}{n^{\alpha}} = \frac{2^{1-\alpha}(1-\alpha)^{\alpha}\pi}{\sin(\pi\alpha)} \quad \text{in probability.}$$

Proof: By Lemma 5.2, it suffices to show that

$$\lim_{n\to\infty} \frac{L_n - L'_n}{n^{\alpha}} = 0 \quad \text{in probability.} \tag{5.6}$$

We have

$$L_n - L'_n = \sum_{k=2}^{m-1} k(g(T_{k-1}) - g(T_k)) \leq \sum_{k=2}^{m-1} k g'(T_{m-1})(T_{k-1} - T_k).$$

Let A be the event that $T_{m-1} \geq 2/(m-1) - 2/n - C/n^{9/8}$, which has probability at least $1-\epsilon$ by Lemma 5.1. There is a positive constant C_2 such that $g'(T_{m-1}) \leq C_2 n^{3\alpha/4}$ on A for all n. Therefore,

$$\begin{aligned}
E[L_n - L'_n | A] &\leq C_2 n^{3\alpha/4} \sum_{k=2}^{m-1} k E[T_{k-1} - T_k] \\
&\leq C_2 n^{3\alpha/4} \sum_{k=2}^{m-1} \frac{2}{k-1} \\
&\leq C_2 n^{3\alpha/4}(1 + \log n).
\end{aligned}$$

Thus, by Markov's Inequality,

$$P(L_n - L'_n > \epsilon n^\alpha) \leq P(A^c) + \frac{E[L_n - L'_n | A]}{\epsilon n^\alpha} \leq \epsilon + \frac{C_2}{\epsilon} n^{-\alpha/4}(1 + \log n),$$

which is less than 2ϵ for sufficiently large n. The result follows. □

Recall that L_n is the sum of the lengths of all branches in the coalescent tree. Also, recall that mutations occur along each branch of the coalescent tree at times of a Poisson process of rate θ. Therefore, if we denote by S_n the number of mutations in the tree, then conditional on L_n, the distribution of S_n is Poisson with mean θL_n. Thus, Lemma 5.3 and Chebyshev's Inequality immediately yield the following result.

Corollary 5.4. *We have*

$$\lim_{n \to \infty} \frac{S_n}{n^\alpha} = \frac{\theta 2^{1-\alpha}(1-\alpha)^\alpha \pi}{\sin(\pi\alpha)} \quad \text{in probability.}$$

Theorem 1.6 now follows from Corollary 5.4 and the next lemma.

Lemma 5.5.

$$\lim_{n \to \infty} \frac{S_n - K_n}{n^\alpha} = 0 \quad \text{in probability.}$$

Proof: Note that if the most recent mutation inherited by two sampled individuals is the same, then all of the mutations inherited by these individuals must be the same. This is because when we follow the two lineages backwards in time, they must coalesce before any mutations are observed. Therefore, each block of the allelic partition Π_n can be associated with a mutation that is the most recent mutation inherited by the individuals in that block, with the possible exception of one block corresponding to individuals with no mutations. It follows that $K_n \leq S_n + 1$.

To get a bound in the other direction, note that the only mutations that are not associated with a block of the allelic partition as above are the mutations that are not the most recent mutation inherited by any individual. We denote the number of such mutations by B_n. Then $K_n \geq S_n - B_n$, so it suffices to show that B_n/n^α converges in probability to zero as $n \to \infty$.

Let R_n denote the number of mutations that occur when the number of lineages is $\lceil n^{3/4} \rceil$ or fewer. Enumerate the remaining mutations in decreasing order of time, so that the first mutation is the most recent one, the second mutation is the second most recent, and so on. Let $R_{k,n}$ denote the number of mutations along the branch of the coalescent tree that we get by starting at the kth mutation and following this lineage back until time

$$g\left(\frac{2}{m-1} - \frac{2}{n} + \frac{C}{n^{9/8}}\right),$$

where C is the constant from Lemma 5.1. Choose $C_3 > \theta 2^{1-\alpha}(1-\alpha)^\alpha \pi/(\sin(\pi\alpha))$. On the event that $T_{m-1} \leq 2/(m-1) - 2/n + C/n^{9/8}$, which has probability at least $1 - \epsilon$ by Lemma 5.1, and on the event that $S_n \leq C_3 n^\alpha$, which has probability tending to one as $n \to \infty$ by Corollary 5.4, we have

$$B_n \leq R_n + \sum_{k=1}^{\lfloor C_3 n^\alpha \rfloor} R_{k,n}. \tag{5.7}$$

Conditional on L_n and L'_n, the distribution of R_n is Poisson with mean $\theta(L_n - L'_n)$. Therefore, by (5.6), R_n/n^α converges to zero in probability as $n \to \infty$. Because mutations occur along each lineage at rate θ, we have for all $k \leq \lfloor C_3 n^\alpha \rfloor$,

$$E[R_{k,n}] \leq \theta g\left(\frac{2}{m-1} - \frac{2}{n} + \frac{C}{n^{9/8}}\right) \leq C_4 n^{-3(1-\alpha)/4}$$

for some positive constant C_4. By summing over k and then applying Markov's Inequality, we get that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{k=1}^{\lfloor C_3 n^\alpha \rfloor} R_{k,n} = 0 \text{ in probability.}$$

Since (5.7) holds with probability at least $1 - 2\epsilon$ for sufficiently large n, the result follows. □

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