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Bounds for scaling exponents for a 1+1 dimensional directed polymer in a Brownian environment

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Abstract. We study the scaling exponents of a 1+1-dimensional directed polymer in a Brownian random environment introduced by O'Connell and Yor. For a version of the model with boundary conditions that are stationary in a space-time sense we identify the exact values of the exponents. For the version without the boundary conditions we get the conjectured upper bounds on the exponents.

1. Introduction

We study the scaling exponents of a directed polymer model in 1+1 dimensions (one space dimension plus time dimension) whose random environment is constructed from Brownian motions. For a positive integer n and an inverse temperature parameter $\beta > 0$, the partition function is defined by

$$Z_n(\beta) = \int_{0<s_1<\cdots<s_{n-1}<n} \exp\bigl[\beta\bigl(B_1(0,s_1) + B_2(s_1,s_2) + \cdots + B_n(s_{n-1},n)\bigr)\bigr]\, ds_{1,n-1} \tag{1.1}$$

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where $\{B_j\}$ are independent one-dimensional standard Brownian motions, $B(s,t) = B(t) - B(s)$, and $ds_{1,n-1} = ds_1 ds_2 \cdots ds_{n-1}$. This model was introduced by O'Connell and Yor (2001) in connection with a related polymer model that they named the *generalized Brownian queue*. Subsequently the exact limiting free energy density $p(\beta)$ was computed by Moriarty and O'Connell (2007). To state their result, recall the gamma function $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx$ and the digamma function $\Psi_0 = \Gamma'/\Gamma$.

Theorem 1.1 (Moriarty and O'Connell, 2007)**.** *For $\beta > 0$ this almost sure limit holds:*

$$\lim_{n\to\infty} n^{-1}\log Z_n(\beta) = p(\beta) = \inf_{t>0}\big\{t\beta^2 - \Psi_0(t)\big\} - 2\log\beta.$$

Moriarty and O'Connell extracted this result with the help of large deviation asymptotics from the generalized Brownian queuing system whose limit is readily computable. In this related model the polymer path is allowed to begin in the infinite past and a decaying exponential factor is included inside the integral to make the partition function converge. We also work with the Brownian queueing system to obtain an estimate on the fluctuations:

Theorem 1.2. *There exist finite, positive β-dependent constants b_0, n_0, C such that for $b \ge b_0$ and $n \ge n_0$*

$$\mathbb{P}\big(\,|\log Z_n(\beta) - np(\beta)| \ge bn^{1/3}\big) \le Cb^{-3/2}.$$

The conjectured behavior for directed polymers is that the order of magnitude of the fluctuations of $\log Z_n(\beta)$ is n^χ and in 1+1 dimensions this exponent takes the value $\chi = 1/3$ at all inverse temperatures $\beta > 0$. Theorem 1.2 gives the expected upper bound on the exponent: $\chi \le 1/3$. In Theorem 2.1 below we give the corresponding upper bound on the fluctuations of the polymer path.

The generalized Brownian queueing system amounts to putting boundary conditions on the polymer (1.1) that are stationary in a natural two-dimensional manner. This stationarity comes from a Burke-type property discovered by O'Connell and Yor (2001), see Lemma 3.2 below. For the model with boundary conditions we identify the exact scaling exponents. Our analysis of these models adapts the steps of the recent work (Seppäläinen, 2010) where a discrete lattice model with analogous properties was discovered and its scaling exponents studied. The roots of the proofs in (Seppäläinen, 2010) can be traced back to the seminal paper (Cator and Groeneboom, 2006). In the context of maximal increasing paths on planar Poisson points, Cator and Groeneboom (2006) were the first to chart a path to the scaling exponents of a two-dimensional growth model without asymptotic analysis of Fredholm determinants.

Suboptimal but still highly nontrivial bounds on scaling exponents in 1+1 dimensional polymer models have been obtained for Brownian polymers in Poisson environments by Comets and Yoshida (2005) and Wüthrich (1998a,b), for Gaussian random walk in a Gaussian environment by Mejane (2004) and Petermann (2000), and for the related model of first passage percolation by Licea et al. (1996) and Newman and Piza (1995).

The overall situation in 1+1 dimensional polymers is now similar to that for two-dimensional directed last-passage percolation models, which are of course closely related as zero-temperature directed polymers. In both areas there is a Brownian model and some particular discrete models that are amenable to explicit computations. Models with general distributions remain beyond the reach of current techniques.

Currently results are farther along for last-passage models: in addition to exponents, explicit Tracy-Widom limit distributions are known. Key results are by Baik et al. (1999), Balázs et al. (2006), Baryshnikov (2001), Cator and Groeneboom (2006), Ferrari and Spohn (2006), Gravner et al. (2001), and Johansson (2000a,b). The connection between last passage models and random matrix theory has been one of the major inspirations of the subject. The recent article (O'Connell, 2009) finds a connection between the Brownian polymer (1.1) and the quantum Toda lattice and proposes this as the possible polymer analogue of the random matrix connection of last-passage percolation.

The Brownian last-passage model arises as the zero temperature limit in the polymer model. With $Z_n(\beta)$ from (1.1),

$$\lim_{\beta\nearrow\infty}\beta^{-1}\log Z_n(\beta)=L_{n,n}\equiv\sup_{0=s_0<s_1<\cdots<s_{n-1}<s_n=n}\sum_{j=1}^{n}B_j(s_{j-1},s_j).$$

We can scale the Brownian motions to the time interval $[0,1]$ so that $L_{n,n}\stackrel{d}{=}\sqrt{n}L_{n,1}$. It has been known for some time that $L_{n,1}$ has the distribution of the largest eigenvalue of an $n\times n$ GUE random matrix (Baryshnikov, 2001; Gravner et al., 2001). So as $n\to\infty$, under the correct scaling $L_{n,n}$ does converge to the Tracy-Widom GUE distribution. If we believe that the finite temperature polymer should behave like the zero temperature model, we have heuristic justification for expecting a Tracy-Widom limit also for the free energy $\log Z_n(\beta)$.

There is one directed polymer model which is essentially solved: the continuum random directed polymer in 1+1 dimension which describes a Brownian path in a white noise environment. The free energy is defined as

$$\log E\left[:\!\exp\!:\left\{-\int_0^T\dot{\mathcal{W}}(t,b(t))dt\right\}\right]$$

where $\dot{\mathcal{W}}$ is a space-time white noise, b is a Brownian motion (or bridge) and $:\!\exp\!:$ is the Wick-ordered exponential. Balázs et al. (2009) determine the exact scaling exponent in the case of a stationary boundary condition using a connection to the KPZ equation and the weakly asymmetric simple exclusion process. Amir et al. (2000) use the same connection to compute the exact distribution of the free energy in the point-to-point setting, and the Tracy-Widom distribution is derived in the appropriate scaling limit. (See also Sasamoto and Spohn, 2010).

Some frequently used notation. We write $f(s,t)=f(t)-f(s)$ for increments, without assuming that $s\le t$. $\mathbb{Z}_+=\{0,1,2,\dots\}$, $\mathbb{N}=\{1,2,3,\dots\}$ and $\mathbb{R}_+=[0,\infty)$. $\overline{X}=X-\mathbb{E}X$ denotes a centered random variable. In general a superscript ω is added to a symbol whenever its dependence on a particular realization of the environment needs to be made explicit.

2. Models and results

We begin with precise definitions of two polymer models, the one already encountered and another one with a boundary. B and $\{B_k:k\in\mathbb{N}\}$ denote independent standard Brownian motions indexed by $\mathbb{R}$. They form the random environment ω under probability measure $\mathbb{P}$.

By Brownian scaling $\{B(ct)\} \overset{d}{=} \{c^{1/2}B(t)\}$ the parameter β can be removed from the exponent in (1.1) and replaced by a parameter that controls the upper limit of integration. This is convenient for us, so instead of $Z_n(\beta)$ we work with the family

$$Z_{j,k}(s,t) = \int\limits_{s<s_j<\cdots<s_{k-1}<t} \exp\bigl[B_j(s,s_j) + B_{j+1}(s_j,s_{j+1}) + \cdots + B_k(s_{k-1},t)\bigr]\,ds_{j,k-1} \tag{2.1}$$

where $1 \le j \le k \in \mathbb{N}$ and $s < t \in \mathbb{R}$. Occasionally we may also write $Z_{(j,k),(s,t)} = Z_{j,k}(s,t)$. The distributional identity is

$$Z_n(\beta) \overset{d}{=} \beta^{-2(n-1)} Z_{1,n}(0, n\beta^2). \tag{2.2}$$

This is the last appearance of the partition function $Z_n(\beta)$ defined by (1.1) in the paper. Similar notation will be used below for other partition functions.

The partition function is the normalizing constant for the quenched polymer distribution $Q_{(j,k),(s,t)}$. This is a probability measure on nondecreasing cadlag paths $x : [s,t] \to \{j, j+1, \dots, k\}$ that go from $x(s) = j$ to $x(t) = k$. We represent these paths in terms of the jump times $s < \sigma_j < \sigma_{j+1} < \cdots < \sigma_{k-1} \le t$ where $x(\sigma_i-) = i < i+1 = x(\sigma_i)$. The measure $Q_{(j,k),(s,t)}$ is defined by

$$E^{Q_{(j,k),(s,t)}} f(\sigma_j,\dots,\sigma_{k-1}) = \frac{1}{Z_{(j,k),(s,t)}} \int\limits_{s<s_j<\cdots<s_{k-1}<t} f(s_j,\dots,s_{k-1}) \times \exp\bigl[B_j(s,s_j) + B_{j+1}(s_j,s_{j+1}) + \cdots + B_k(s_{k-1},t)\bigr]\,ds_{j,k-1}. \tag{2.3}$$

This measure is called quenched because the environment of Brownian motions is fixed. Integrating away the environment gives the annealed expectation $E_{(j,k),(s,t)}(\cdot) = \mathbb{E}E^{Q_{(j,k),(s,t)}}(\cdot)$.

In addition to the digamma function we also need its derivative, the trigamma function $\Psi_1 = \Psi_0'$. Ψ_0 is concave and increasing and Ψ_1 is positive, convex and strictly decreasing with $\Psi_1(0+) = \infty$ and $\Psi_1(\infty) = 0$. Theorem 1.1 is equivalent to the statement

$$\lim_{n\to\infty} n^{-1} \log Z_{1,n}(0, n\tau) = \Psi_1(\theta)\theta - \Psi_0(\theta) \quad \mathbb{P}\text{-a.s.} \tag{2.4}$$

where $\tau > 0$ and θ is the unique value such that $\Psi_1(\theta) = \tau$. Theorem 1.2 will be proved in Section 7 in terms of $Z_{1,n}(0, n\tau)$. In conjunction with Theorem 1.2 goes an upper bound on the fluctuations of the path, also proved in Section 7.

Theorem 2.1. *Let $\tau > 0$ and $0 < \gamma < 1$. Then for all large enough n and b*

$$P_{(1,n),(0,n\tau)}\bigl(|\sigma_{\lfloor n\gamma\rfloor} - n\gamma\tau| > bn^{2/3}\bigr) \le C(\tau)b^{-3}. \tag{2.5}$$

Theorem 2.1 says that the path stays close to the diagonal of the rectangle $[1,n] \times [0, n\tau]$, and typical fluctuations away from the diagonal have order of magnitude at most $n^{2/3}$. This gives an upper bound $\zeta \le 2/3$ for the second basic scaling exponent ζ which describes the fluctuations of the polymer path.

These are the results for the polymer without a boundary, and we turn to discuss the model with boundary. For this model the upper bounds of Theorems 1.2 and

2.1 are combined with matching lower bounds, so we have the precise values of the scaling exponents. An additional parameter $\theta > 0$ is introduced in this model.

The partition function is

$$Z_n^\theta(t) = Z_{n,t}^\theta = \int_{-\infty < s_0 < s_1 < \cdots < s_{n-1} < t} \exp\bigl[-B(s_0) + \theta s_0 + B_1(s_0, s_1) + B_2(s_1, s_2) + \cdots + B_n(s_{n-1}, t)\bigr]\, ds_{0,n-1}. \tag{2.6}$$

The quenched polymer measure $Q_{n,t}^\theta$ lives on nondecreasing cadlag paths

$$x : (-\infty, t] \to \{0, 1, \ldots, n\} \text{ that go from } x(-\infty) = 0 \text{ to } x(t) = n.$$

Again we represent these in terms of jump times $-\infty < \sigma_0 < \sigma_1 < \cdots < \sigma_{n-1} \le t$ where $x(\sigma_i-) = i < i + 1 = x(\sigma_i)$. The measure is defined by

$$E^{Q_{n,t}^\theta} f(\sigma_0, \sigma_1, \ldots, \sigma_{n-1}) = \frac{1}{Z_n^\theta(t)} \int\limits_{-\infty < s_0 < \cdots < s_{n-1} < t} f(s_0, s_1, \ldots, s_{n-1}) \times \exp\bigl[-B(s_0) + \theta s_0 + B_1(s_0, s_1) + B_2(s_1, s_2) + \cdots + B_n(s_{n-1}, t)\bigr]\, ds_{0,n-1}. \tag{2.7}$$

Annealed probability and expectation are denoted by $P_{n,t}^\theta(\cdot) = \mathbb{E}Q_{n,t}^\theta(\cdot)$ and $E_{n,t}^\theta(\cdot) = \mathbb{E}E^{Q_{n,t}^\theta}(\cdot)$, and simply $E(\cdot)$ when the parameters are understood.

Notational remark. To simplify notation we shall not consistently carry the superscript θ in the notation for the objects of the polymer model with boundary. The notational distinction between the two models is that the model without boundary has two space-time parameters represented by $((j, k), (s, t))$ in definitions (2.1) and (2.3), while the model with boundary has only a single space-time parameter, namely (n, t) in definitions (2.6) and (2.7). When dependence on the environment ω needs to be displayed explicitly, ω is added as a superscript, as for example in $Q_{n,t}^{\theta,\omega}$.

The boundary conditions render the model stationary in a sense made precise in Theorem 3.3. As a consequence some explicit computations can be performed: from Theorems 3.3 and 3.6 we obtain

$$\mathbb{E}(\log Z_n^\theta(t)) = -n\Psi_0(\theta) + \theta t$$

and

$$\mathbb{V}\text{ar}(\log Z_n^\theta(t)) = n\Psi_1(\theta) - t + 2E_{n,t}^\theta(\sigma_0^+). \tag{2.8}$$

As we take the size of the polymer to infinity, the interesting exponents appear when the endpoint follows approximately a *characteristic direction* specified by the parameter θ. As t and n become large, they will be assumed to satisfy

$$|t - n\Psi_1(\theta)| \le An^{2/3} \text{ for a constant } 0 \le A < \infty. \tag{2.9}$$

The purpose of this assumption is to kill the first two terms of the variance formula in (2.8). The first theorem says that the exponent $\chi = 1/3$ describes the order of fluctuations of $\log Z_n^\theta(t)$.

Theorem 2.2. *Fix $0 < A < \infty$. Then there exist constants $0 < C_1 < C_2 < \infty$ that depend on (θ, A) such that, for all $t > 0$ and $n \ge 1$ that satisfy (2.9),*

$$C_1 n^{2/3} \le \mathbb{V}\text{ar}(\log Z_n^\theta(t)) \le C_2 n^{2/3}. \tag{2.10}$$

The second theorem says that under the annealed distribution the path stays close to the diagonal of the rectangle $[0,n]\times[0,t]$, and $n^{2/3}$ is the correct order of typical fluctuations.

Theorem 2.3. *Fix $0<A<\infty$ and consider $t>0$ and $n\ge 1$ that satisfy (2.9). Fix $0\le\gamma<1$. Then for large enough n and b*

$$P^{\theta}_{n,t}(|\sigma_{\lfloor\gamma n\rfloor}-\gamma t|>bn^{2/3})\le Cb^{-3}. \tag{2.11}$$

For any $\varepsilon>0$ there exists $\delta>0$ such that

$$\overline{\lim_{n\to\infty}}\, P^{\theta}_{n,t}(|\sigma_{\lfloor\gamma n\rfloor}-\gamma t|\le\delta n^{2/3})\le\varepsilon. \tag{2.12}$$

Variable σ_0 is a special case of Theorem 2.3, but in fact controlling this case turns out to be the key to *both* Theorems 2.2 and 2.3. Let us also mention that our arguments give the moment bound $E^{\theta}_{n,t}(|\sigma_0|^p)\le Cn^{2p/3}$ for $1\le p<3$.

3. Properties of the model with boundary

Following O'Connell and Yor (2001) introduce the following processes to render the role of boundary conditions clearer in (2.6): $Y_0(t)=B(t)$, and then define inductively for $k\in\mathbb{N}$

$$r_k(t)=\log\int_{-\infty}^{t}e^{Y_{k-1}(s,t)-\theta(t-s)+B_k(s,t)}\,ds \tag{3.1}$$

and

$$\begin{aligned} Y_k(t)&=Y_{k-1}(t)+r_k(0)-r_k(t),\\ X_k(t)&=B_k(t)+r_k(0)-r_k(t). \end{aligned} \tag{3.2}$$

From the definitions and a simple induction argument follow the equations

$$\begin{aligned} \sum_{k=1}^{n}r_k(t)&=\log\int_{-\infty<s_0<s_1\cdots<s_{n-1}<t}\exp\bigl[B(s_0,t)-\theta(t-s_0)\\ &\qquad\qquad+B_1(s_0,s_1)+B_2(s_1,s_2)+\cdots+B_n(s_{n-1},t)\bigr]\,ds_{0,n-1}\\ &=B(t)-\theta t+\log Z^{\theta}_n(t) \end{aligned} \tag{3.3}$$

and

$$Z^{\theta}_n(t)=\int_0^t\exp[-B(s)+\theta s]Z_{1,n}(s,t)\,ds\;+\;\sum_{j=1}^{n}\Bigl(\prod_{k=1}^{j}e^{r_k(0)}\Bigr)Z_{j,n}(0,t). \tag{3.4}$$

If we define also

$$Z^{\theta}_0(t)=\exp[-B(t)+\theta t] \tag{3.5}$$

then the above can be expressed in the form

$$Z^{\theta}_n(t)=\int_0^t Z^{\theta}_0(s)Z_{1,n}(s,t)\,ds\;+\;\sum_{j=1}^{n}Z^{\theta}_j(0)Z_{j,n}(0,t). \tag{3.6}$$

Combining (3.3) with (3.2) gives the space and time increments

$$\log Z^{\theta}_n(t)-\log Z^{\theta}_{n-1}(t)=r_n(t)\qquad\text{for } n\in\mathbb{N},\ t\in\mathbb{R}_+ \tag{3.7}$$

and

$$\log Z^{\theta}_n(t)-\log Z^{\theta}_n(s)=\theta(t-s)-Y_n(s,t)\qquad\text{for } n\in\mathbb{Z}_+,\ s<t\in\mathbb{R}_+. \tag{3.8}$$

Remark 3.1. The density of $(\sigma_0,\dots,\sigma_{n-1})$ under $Q_{n,t}$ can also be written as

$$\frac{1}{\widehat{Z}_n^\theta(t)}\exp\big[\widehat{B}(s_0,t)+\widehat{B}_1(s_0,s_1)+\widehat{B}_2(s_1,s_2)+\cdots+\widehat{B}_n(s_{n-1},t)\big] \\ \times \mathbf{1}\{s_0<\cdots<s_{n-1}<t\} \tag{3.9}$$

where $\widehat{B}(u)=B(u)-\theta u/2$ (and similarly for $\widehat{B}_k$) and $\widehat{Z}_n^\theta(t)=Z_n^\theta(t)\exp(B(t)-\theta t)$. From this representation and the stationarity of Brownian increments it is clear that

$$E^{Q_{n,t}}f(\sigma_0,\sigma_1,\dots,\sigma_{n-1})\overset{d}{=}E^{Q_{n,0}}f(t+\sigma_0,t+\sigma_1,\dots,t+\sigma_{n-1}). \tag{3.10}$$

3.1. *Burke property.* This lemma summarizes the Burke property of (O'Connell and Yor, 2001, Thm. 5).

Lemma 3.2 (O'Connell and Yor, 2001)**.** *Let B and C be independent standard Brownian motions indexed by $\mathbb{R}$ and $\theta>0$ a fixed constant. For $t\in\mathbb{R}$ set*

$$r(t)=\log\int_{-\infty}^t \exp\left(B(s,t)+C(s,t)-\theta(t-s)\right)ds,$$
$$f(t)=B(t)+r(0)-r(t),\qquad g(t)=C(t)+r(0)-r(t).$$

Then f and g are independent standard Brownian motions indexed by $\mathbb{R}$ and for each $t\ge 0$ the processes $\{(f(s),g(s)):s\le t\}$ and $\{r(s):s\ge t\}$ are independent. Moreover the following identity holds almost surely:

$$r(t)=\log\int_t^\infty \exp\left(f(t,s)+g(t,s)+\theta(t-s)\right)ds. \tag{3.11}$$

Although identity (3.11) is not stated explicitly in Theorem 5 of O'Connell and Yor (2001) it is contained in the proof. In the next theorem we generalize the Burke property in a form suitable for our use.

Theorem 3.3. *Let $n\in\mathbb{N}$ and $0\le s_n\le s_{n-1}\le\cdots\le s_1<\infty$. Then over the index j the following random variables and processes are all mutually independent:*

$$r_j(s_j)\ \ \textit{and}\ \ \{X_j(s):s\le s_j\}\ \ \textit{for}\ \ 1\le j\le n,\quad \{Y_n(s):s\le s_n\},$$
$$\textit{and}\quad \{Y_j(s_{j+1},s):s_{j+1}\le s\le s_j\}\ \ \textit{for}\ \ 1\le j\le n-1.$$

Furthermore, the X_j and Y_j processes are standard Brownian motions, and $r_j(s_j)\overset{d}{=}-\log\eta$ with $\eta\sim Gamma(\theta,1)$.

Proof: We use induction on n. For $n=1$ the statement follows from Lemma 3.2 together with Dufresne's identity (Dufresne, 2001, Cor. 4) which gives the distribution of $r_1(t)$.

We now assume that the statement is true for $n-1$ and prove it for n. From definitions (3.1)–(3.2), $\{r_n(s),Y_n(s),X_n(s):s\le s_n\}$ is a function of $\{Y_{n-1}(s):s\le s_n\}$ and an independent Brownian motion B_n. This means that $\{r_n(s),Y_n(s),X_n(s):s\le s_n\}$ is independent of the processes $\{X_j(s):s\le s_j\}$, $\{Y_j(s_{j+1},s):s_{j+1}\le s\le s_j\}$ for $1\le j\le n-1$, and random variables $\{r_j(s_j):1\le j\le n-1\}$. Besides the induction hypothesis we also used that $\{Y_{n-1}(s_n,s),s\ge s_n\}$ is independent of $\{Y_{n-1}(s):s\le s_n\}$. That $\{Y_n(s),X_n(s):s\le s_n\}$ and $r_n(s_n)$ have the right joint distribution again follows from Lemma 3.2 and Dufresne's identity. $\square$

Applying the shift $S_u f(t) = f(u+t) - f(u)$ that preserves the distributions of the Brownian motions shows that $r_k(t) = r_k(0) \circ S_t$ is a stationary process. We also note that $\mathbb{E}(r_k(t)) = -\Psi_0(\theta)$ and $\mathbb{V}\mathrm{ar}(r_k(t)) = \Psi_1(\theta)$.

3.2. *Reversal.* Fix $T \in \mathbb{R}$ and $n \in \mathbb{N}$. Define the following new processes:

$$\begin{aligned} Y_j^*(s) &= Y_{n-j}(T) - Y_{n-j}(T-s), \qquad 0 \le j \le n, && (3.12)\\ B_j^*(s) &= X_{n+1-j}(T) - X_{n+1-j}(T-s), \qquad 1 \le j \le n, && (3.13)\\ r_j^*(s) &= r_{n+1-j}(T-s), \qquad 1 \le j \le n, && (3.14)\\ X_j^*(s) &= B_{n+1-j}(T) - B_{n+1-j}(T-s), \qquad 1 \le j \le n. && (3.15) \end{aligned}$$

Define the *dual environment* by $\omega^* = (Y_0^*, B_j^* : 1 \le j \le n)$.

Theorem 3.4. *Fix $n \in \mathbb{N}$. We have the following equality in distribution for the processes on $\mathbb{R}$:*

$$\begin{aligned} &\{Y_k : 0 \le k \le n;\ X_j, B_j, r_j : 1 \le j \le n\} \\ &\quad \stackrel{d}{=} \left\{Y_k^* : 0 \le k \le n;\ X_j^*, B_j^*, r_j^* : 1 \le j \le n\right\}. \end{aligned}$$

In particular, the dual environment $\omega^* = (Y_0^*, B_j^* : 1 \le j \le n)$ has the same distribution as the original environment $\omega = (B, B_j : 1 \le j \le n)$.

Proof: By (3.11) from Lemma 3.2

$$r_k(t) = \log \int_t^\infty \exp(Y_k(t,s) + \theta(t-s) + X_k(t,s))ds. \tag{3.16}$$

Definitions (3.12)–(3.15) together with (3.2) and (3.16) give the identities

$$\begin{aligned} r_k^*(t) &= \log \int_{-\infty}^t \exp(Y_{k-1}^*(s,t) - \theta(t-s) + B_k^*(s,t))\, ds, \\ Y_k^*(t) &= Y_{k-1}^*(t) + r_k^*(0) - r_k^*(t), \\ X_k^*(t) &= B_k^*(t) + r_k^*(0) - r_k^*(t). \end{aligned} \tag{3.17}$$

Theorem 3.3 tells us that $\{Y_n, X_j, 1 \le j \le n\}$ are independent Brownian motions on $(-\infty, t]$ for any $t > 0$, hence over all of $\mathbb{R}$. Consequently by (3.12)–(3.13) processes $\{Y_0^*, B_j^*, 1 \le j \le n\}$ have the same distribution as $\{Y_0, B_j, 1 \le j \le n\}$. The theorem follows because (3.17) defines the same recursions as (3.1)–(3.2). □

We define the dual quenched measure Q_n^* on non-decreasing cadlag paths $x : \mathbb{R}_+ \to \{0, 1, \ldots, n\}$ with $x(0) = 0$ and $x(\infty) = n$. These paths can be represented by jump times $0 < \sigma_1^* < \cdots < \sigma_n^*$ defined by $x(\sigma_j^*-) = j - 1 < j = x(\sigma_j^*)$. The dual measure Q_n^* is defined by

$$\begin{aligned} E^{Q_n^*}\left[f(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)\right] &= \frac{1}{Z_n^*} \int_{0<s_1<\cdots<s_n<\infty} f(s_1, s_2, \ldots, s_n) \\ &\quad \times \exp\bigl[X_1(0,s_1) + X_2(s_1,s_2) + \\ &\qquad \cdots + X_n(s_{n-1}, s_n) + Y_n(0, s_n) - \theta s_n\bigr]\, ds_{1,n}. \end{aligned} \tag{3.18}$$

Because of the shifting described in (3.10) we only need the measure Q_n^* instead of a family indexed also by t. The dual measure is naturally connected with the representation of $Q_{n,t}$ given in Remark 3.1. The next lemma is proved by a straightforward change of variables in the integrals. Recall the definition of $\widehat{Z}_n^\theta(t)$ from (3.9).

Lemma 3.5. *Fix $t \in \mathbb{R}$ and $n \in \mathbb{N}$ and for each ω define the dual environment ω^* with $T = t$. Then $Z_n^{*,\omega} = \widehat{Z}_n^{\theta,\omega^*}(t)$, and*

$$E^{Q_n^{*,\omega}}\big[f(\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)\big] = E^{Q_{n,t}^{\omega^*}}\big[f(t-\sigma_{n-1}, t-\sigma_{n-2}, \dots, t-\sigma_0)\big].$$

Consequently, by Theorem 3.4,

$$E^{Q_n^*}\big[f(\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)\big] \overset{d}{=} E^{Q_{n,t}}\big[f(t-\sigma_{n-1}, t-\sigma_{n-2}, \dots, t-\sigma_0)\big].$$

3.3. *Variance identity.*

Theorem 3.6. *For $\theta > 0$ and $(t, n) \in (0, \infty) \times \mathbb{N}$*

$$\begin{aligned} \mathbb{V}\mathrm{ar}[\log Z_n^\theta(t)] &= n\Psi_1(\theta) - t + 2E_{n,t}^\theta(\sigma_0^+) \\ &= -n\Psi_1(\theta) + t + 2E_{n,t}^\theta(\sigma_0^-). \end{aligned} \tag{3.19}$$

Remark 3.7. Adding the two equations gives $\mathbb{V}\mathrm{ar}[\log Z_n^\theta(t)] = E_{n,t}^\theta|\sigma_0|$ while subtracting them yields $E_{n,t}^\theta(\sigma_0) = t - n\Psi_1(\theta)$.

Proof of Theorem 3.6: While we keep θ fixed we simplify the notation to $Z_n(t) = Z_n^\theta(t)$.

The proof begins the same way as the proof of Theorem 3.7 in (Seppäläinen, 2010). (The idea was originally learned from the proof of Theorem 2.1 in (Cator and Groeneboom, 2006).) Abbreviate the increments with reference to compass directions:

$$\begin{aligned} \mathcal{N} &= \log Z_n(t) - \log Z_n(0), \quad \mathcal{S} = \log Z_0(t) = \theta t - B(t), \\ \mathcal{E} &= \log Z_n(t) - \log Z_0(t), \quad \mathcal{W} = \log Z_n(0) = \sum_{j=1}^n r_j(0). \end{aligned}$$

The south and west increments are given by the boundary conditions as indicated above, while the east and north increments are computed from (3.4). Then

$$\begin{aligned} \mathbb{V}\mathrm{ar}\big[\log Z_n(t)\big] &= \mathbb{V}\mathrm{ar}(\mathcal{W} + \mathcal{N}) = \mathbb{V}\mathrm{ar}(\mathcal{W}) + \mathbb{V}\mathrm{ar}(\mathcal{N}) + 2\mathbb{C}\mathrm{ov}(\mathcal{W}, \mathcal{N}) \\ &= \mathbb{V}\mathrm{ar}(\mathcal{W}) + \mathbb{V}\mathrm{ar}(\mathcal{N}) + 2\mathbb{C}\mathrm{ov}(\mathcal{S} + \mathcal{E} - \mathcal{N}, \mathcal{N}) \\ &= \mathbb{V}\mathrm{ar}(\mathcal{W}) - \mathbb{V}\mathrm{ar}(\mathcal{N}) + 2\mathbb{C}\mathrm{ov}(\mathcal{S}, \mathcal{N}) \\ &= n\Psi_1(\theta) - t + 2\mathbb{C}\mathrm{ov}(\mathcal{S}, \mathcal{N}). \end{aligned} \tag{3.20}$$

The third line comes from the independence of $\mathcal{E}$ and $\mathcal{N}$ while the last equality came from (3.8) and Theorem 3.3. Substituting $\mathcal{N}$ instead of $\mathcal{W}$ in the covariance in the second line of (3.20) leads to the formula

$$\mathbb{V}\mathrm{ar}\big[\log Z_n(t)\big] = -n\Psi_1(\theta) + t + 2\mathbb{C}\mathrm{ov}(\mathcal{E}, \mathcal{W}). \tag{3.21}$$

Equations (3.20) and (3.21) give the two lines of (3.19) once we evaluate the covariances. We begin with (3.20).

We need to vary separately the parameters of the boundary conditions on the x and y axes. So we rename the parameter on the y-axis as λ, and rewrite the partition function with boundaries as follows:

$$
\begin{aligned}
Z_n(t) = \int_{0<s_0<s_1\cdots<s_{n-1}<t} &\exp\bigl[-B(s_0)+\theta s_0 \\
&\quad + B_1(s_0,s_1)+B_2(s_1,s_2)+\cdots+B_n(s_{n-1},t)\bigr]\,ds_{0,n-1} \\
+ \int_{-\infty<s_0<s_1\cdots<s_{n-1}<t} &\mathbf{1}\{s_0<0\}\exp\bigl[-B(s_0)+\lambda s_0 \\
&\quad + B_1(s_0,s_1)+B_2(s_1,s_2)+\cdots+B_n(s_{n-1},t)\bigr]\,ds_{0,n-1}.
\end{aligned}
\tag{3.22}
$$

Next we argue that

$$
\mathbb{E}(\mathcal{N}|\mathcal{S}=x) \text{ does not depend on } \theta. \tag{3.23}
$$

Indeed, if we condition $\{h(s)=\theta s-B(s), 0\le s\le t\}$ on the event $h(t)=x$ then this (as a process) has the same distribution as $\theta s-\tilde{B}_s$ where $\tilde{B}_s$ is a BM conditioned to hit $y=\theta t-x$ at t. It is known that the latter has the same distribution as $B_s-sB_t/t+sy/t$ which means that the conditional distribution of $\{h(s), 0\le s\le t\}$ is the same as

$$
\theta s-\left(B_s-\frac{sB_t}{t}+\frac{s(\theta t-x)}{t}\right)=\frac{s}{t}B_t-B_s+\frac{sx}{t}
$$

which does not depend on θ.

The density of $\mathcal{S}$ is $f_\theta(x)=(2\pi t)^{-1/2}\exp(-\frac{1}{2t}(x-\theta t)^2)$. Utilizing (3.23),

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\mathbb{E}(\mathcal{N}) &= \frac{\partial}{\partial\theta}\int_{\mathbb{R}}\mathbb{E}(\mathcal{N}\,|\,\mathcal{S}=x)f_\theta(x)\,dx=\int_{\mathbb{R}}\mathbb{E}(\mathcal{N}\,|\,\mathcal{S}=x)\frac{\partial f_\theta(x)}{\partial\theta}\,dx \\
&= \int_{\mathbb{R}}\mathbb{E}(\mathcal{N}\,|\,\mathcal{S}=x)(x-\theta t)f_\theta(x)\,dx \\
&= \mathbb{E}(\mathcal{N}\mathcal{S})-\mathbb{E}(\mathcal{N})\mathbb{E}(\mathcal{S})=\mathbb{C}\mathrm{ov}(\mathcal{N},\mathcal{S}).
\end{aligned}
\tag{3.24}
$$

On the other hand, utilizing (3.22),

$$
\begin{aligned}
&\frac{\partial}{\partial\theta}\mathbb{E}(\mathcal{N})=\mathbb{E}\Big[\frac{\partial}{\partial\theta}\log Z_n(t)\Big] \\
&=\mathbb{E}\Big[\frac{1}{Z_n(t)}\int_{0<s_0<s_1\cdots<s_{n-1}<t} s_0\,\exp\bigl\{-B(s_0)+\theta s_0 \\
&\qquad + B_1(s_0,s_1)+B_2(s_1,s_2)+\cdots+B_n(s_{n-1},t)\bigr\}\,ds_{0,n-1}\Big] \\
&=E(\sigma_0^+).
\end{aligned}
\tag{3.25}
$$

Combining (3.24), (3.24) and (3.25) gives the first line of (3.19).

Proof of the second line of (3.19) proceeds analogously. $\mathbb{E}(\mathcal{E}\,|\,\mathcal{W}=x)$ does not depend on λ, and so the analogue of computation (3.24) gives $\partial_\lambda\mathbb{E}(\mathcal{E})=-\mathbb{C}\mathrm{ov}(\mathcal{E},\mathcal{W})$. Then from (3.22), $\partial_\lambda\mathbb{E}(\mathcal{E})=\mathbb{E}(\partial_\lambda\log Z_n(t))=E(\sigma_0\mathbf{1}\{\sigma_0<0\})$. ☐

3.4. *Comparison lemma.* We find it useful here to augment the family $Z_{j,k}(s,t)$ defined for $j\ge 1$ by (2.1) by introducing $Z_{0,k}(t)=Z_{0,k}(0,t)$. These will be defined not exactly consistently with (2.1), but in a manner that gives us inequalities between ratios of partition functions. For $k\in\mathbb{N}$ and $t\in\mathbb{R}_+$ define

$$
Z_{0,0}(t)=e^{-B(t)} \tag{3.26}
$$

and

$$Z_{0,k}(t) = \int\limits_{0<s_0<\cdots<s_{k-1}<t} \exp\bigl[-B(s_0) + B_1(s_0,s_1) \\ + B_2(s_1,s_2) + \cdots + B_k(s_{k-1},t)\bigr]\, ds_{0,k-1}. \tag{3.27}$$

For $n \in \mathbb{N}$ and events D on the paths we write $Z^\theta_{n,t}(D) = Z^\theta_{n,t} Q^\theta_{n,t}(D)$ for the unnormalized quenched measure. It is also convenient to set, for $A \subseteq \mathbb{R}$,

$$Z^\theta_{0,t}(\sigma_0 \in A) = \mathbf{1}_{A\cap\mathbb{R}_+}(t)\exp[-B(t) + \theta t]. \tag{3.28}$$

Lemma 3.8. *Let $\theta > 0$. For $0 < s < t$ and $n \in \mathbb{Z}_+$*

$$\frac{Z^\theta_{n+1,t}(\sigma_0 > 0)}{Z^\theta_{n,t}(\sigma_0 > 0)} \le \frac{Z_{0,n+1}(t)}{Z_{0,n}(t)} \le \frac{Z^\theta_{n+1,t}(\sigma_0 < 0)}{Z^\theta_{n,t}(\sigma_0 < 0)} \tag{3.29}$$

and

$$\frac{Z^\theta_{n,t}(\sigma_0 > 0)}{Z^\theta_{n,s}(\sigma_0 > 0)} \ge \frac{Z_{0,n}(t)}{Z_{0,n}(s)} \ge \frac{Z^\theta_{n,t}(\sigma_0 < 0)}{Z^\theta_{n,s}(\sigma_0 < 0)}. \tag{3.30}$$

The second inequality of (3.30) makes sense only for $n \ge 1$.

Proof: We check the cases that initialize the inductive proofs. For (3.29)

$$\frac{Z^\theta_{1,t}(\sigma_0 > 0)}{Z^\theta_{0,t}(\sigma_0 > 0)} = \frac{\int_0^t e^{-B(s)+\theta s+B_1(s,t)}\,ds}{e^{-B(t)+\theta t}} = \int_0^t e^{B(s,t)+\theta(s-t)+B_1(s,t)}\,ds \\ \le \int_0^t e^{B(s,t)+B_1(s,t)}\,ds = \frac{Z_{0,1}(t)}{Z_{0,0}(t)} < \infty = \frac{Z^\theta_{1,t}(\sigma_0 < 0)}{Z^\theta_{0,t}(\sigma_0 < 0)}.$$

For the first part of (3.30)

$$\frac{Z^\theta_{0,t}(\sigma_0 > 0)}{Z^\theta_{0,s}(\sigma_0 > 0)} = e^{-B(s,t)+\theta(t-s)} > e^{-B(s,t)} = \frac{Z_{0,0}(t)}{Z_{0,0}(s)}$$

and for the second part (now for $n = 1$)

$$\frac{Z_{0,1}(t)}{Z_{0,1}(s)} = e^{B_1(s,t)} + \frac{\int_s^t e^{-B(u)+B_1(u,t)}\,du}{\int_0^s e^{-B(u)+B_1(u,s)}\,du} > e^{B_1(s,t)} = \frac{Z^\theta_{1,t}(\sigma_0 < 0)}{Z^\theta_{1,s}(\sigma_0 < 0)}.$$

Next the induction steps. We make use of the decomposition (for $0 \le s < t$)

$$Z^\theta_{n,t}(\sigma_0 \in A) = Z^\theta_{n,s}(\sigma_0 \in A)e^{B_n(s,t)} + \int_s^t Z^\theta_{n-1,u}(\sigma_0 \in A)e^{B_n(u,t)}\,du$$

valid for $n \in \mathbb{N}$ and for both $A = (-\infty, 0)$ and $A = (0, \infty)$. For $n = 1$ convention (3.28) is used on the right-hand side.

We begin with the first inequalities in (3.29)–(3.30). Assume the first inequality of both (3.29) and (3.30) holds for $n-1$. We verify first (3.30) for n.

$$\begin{aligned}
\frac{Z^\theta_{n,t}(\sigma_0>0)}{Z^\theta_{n,s}(\sigma_0>0)} &= e^{B_n(s,t)} + \int_s^t \frac{Z^\theta_{n-1,u}(\sigma_0>0)}{Z^\theta_{n,s}(\sigma_0>0)}\, e^{B_n(u,t)}\,du \\
&= e^{B_n(s,t)} + \int_s^t \frac{Z^\theta_{n-1,u}(\sigma_0>0)}{Z^\theta_{n-1,s}(\sigma_0>0)}\cdot\frac{Z^\theta_{n-1,s}(\sigma_0>0)}{Z^\theta_{n,s}(\sigma_0>0)}\, e^{B_n(u,t)}\,du \\
&\ge e^{B_n(s,t)} + \int_s^t \frac{Z_{0,n-1}(u)}{Z_{0,n-1}(s)}\cdot\frac{Z_{0,n-1}(s)}{Z_{0,n}(s)}\, e^{B_n(u,t)}\,du \\
&= \frac{Z_{0,n}(t)}{Z_{0,n}(s)}
\end{aligned}$$

and the inequality came from the induction assumption.

Deriving the first inequality of (3.29) for n requires an extra step. First decompose and use (3.30) for n that we just proved:

$$\begin{aligned}
\frac{Z^\theta_{n+1,t}(\sigma_0>0)}{Z^\theta_{n,t}(\sigma_0>0)} &= \frac{Z^\theta_{n+1,s}(\sigma_0>0)}{Z^\theta_{n,t}(\sigma_0>0)}\, e^{B_n(s,t)} + \int_s^t \frac{Z^\theta_{n,u}(\sigma_0>0)}{Z^\theta_{n,t}(\sigma_0>0)}\, e^{B_n(u,t)}\,du \\
&= \frac{Z^\theta_{n+1,s}(\sigma_0>0)}{Z^\theta_{n,s}(\sigma_0>0)}\cdot\frac{Z^\theta_{n,s}(\sigma_0>0)}{Z^\theta_{n,t}(\sigma_0>0)}\, e^{B_n(s,t)} + \int_s^t \frac{Z^\theta_{n,u}(\sigma_0>0)}{Z^\theta_{n,t}(\sigma_0>0)}\, e^{B_n(u,t)}\,du \\
&\le \frac{Z^\theta_{n+1,s}(\sigma_0>0)}{Z^\theta_{n,s}(\sigma_0>0)}\cdot\frac{Z_{0,n}(s)}{Z_{0,n}(t)}\, e^{B_n(s,t)} + \int_s^t \frac{Z_{0,n}(u)}{Z_{0,n}(t)}\, e^{B_n(u,t)}\,du \\
&\equiv \frac{Z^\theta_{n+1,s}(\sigma_0>0)}{Z^\theta_{n,s}(\sigma_0>0)}\, a(s,t) + \int_s^t a(u,t)\,du.
\end{aligned}$$

The last equality defines the continuous function $a(s,t)$. The same steps applied to the middle member of (3.29) gives an equality

$$\frac{Z_{0,n+1}(t)}{Z_{0,n}(t)} = \frac{Z_{0,n+1}(s)}{Z_{0,n}(s)}\, a(s,t) + \int_s^t a(u,t)\,du$$

from which

$$\frac{Z^\theta_{n+1,t}(\sigma_0>0)}{Z^\theta_{n,t}(\sigma_0>0)} - \frac{Z_{0,n+1}(t)}{Z_{0,n}(t)} \;\le\; a(s,t)\biggl(\frac{Z^\theta_{n+1,s}(\sigma_0>0)}{Z^\theta_{n,s}(\sigma_0>0)} - \frac{Z_{0,n+1}(s)}{Z_{0,n}(s)}\biggr). \tag{3.31}$$

Each of the three terms on the right-hand side above vanishes as we take $s \searrow 0$. We have proved that the first inequalities in (3.29)–(3.30) hold for n.

To derive the second inequalities in (3.29)–(3.30) for n the induction proofs work the same way: once $(\sigma_0>0)$ has been replaced by $(\sigma_0<0)$ all inequalities in the above calculations have to be reversed. In (3.31) on the right the ratio $Z^\theta_{n+1,s}(\sigma_0<0)/Z^\theta_{n,s}(\sigma_0<0)$ does not vanish as $s\searrow 0$ but it stays bounded a.s. and again $a(0+,t)=0$ takes the entire right-hand side to 0. $\square$

4. Upper bound for the variance

The parameter $\theta\in(0,\infty)$ can be considered fixed throughout. The constants such as $C(\theta)$ that depend on θ and appear in all our statements are locally bounded functions of θ. Sometimes we will suppress the dependence on θ and the constants in the intermediate estimates may change from line to line. Recall also the key assumption (2.9) on $t > 0$ and $n \in \mathbb{N}$, namely that $|t - n\Psi_1(\theta)| \le An^{2/3}$. We develop the upper bounds so that the effect of the constant A is explicitly present.

Theorem 4.1. *Fix $0 < A < \infty$. Then there exists a constant $C(\theta) < \infty$ such that, for all $t > 0$ and $n \ge 1$ that satisfy assumption (2.9),*

$$\mathbb{V}\mathrm{ar}(\log Z_n^\theta(t)) \le (9A + C(\theta))n^{2/3}. \tag{4.1}$$

The value 9 in the statement has no particular meaning, except as a constant independent of θ. As a first step we give a bound on the tails of $\sigma_0^\pm$.

Lemma 4.2. *Let $t > 0$ and $n \ge 1$ satisfy (2.9), and fix $K > 0$. Then we can fix $C = C(\theta) < \infty$ large enough and $s = s(K, \theta) > 0$ small enough such that, for $3An^{2/3} \le u \le Kn$, we have the bound*

$$\mathbb{P}(Q_{n,t}(\sigma_0^\pm \ge u) \ge e^{-su^2/n}) \le \frac{Cn^2}{u^4} E_{n,t}(\sigma_0^\pm) + \frac{Cn^2}{u^3}. \tag{4.2}$$

Proof: We first consider the proof for σ^+. Set $\lambda = \theta + \frac{bu}{n}$ where $0 < b < 1$. In the course of the proof b will be chosen small. Superscripts θ and λ indicate which parameter is used. We have

$$\begin{aligned}
Q_{n,t}^\theta(\sigma_0^+ \ge u) &= \frac{1}{Z_n^\theta(t)} \int_{-\infty < s_0 < \cdots < s_{n-1} < t} \mathbf{1}(s_0 \ge u) \\
&\qquad \times \exp\big[-B(s_0) + \theta s_0 + B_1(s_0, s_1) + \cdots + B_n(s_{n-1}, t)\big]\, ds_{0,n-1} \\
&\le \frac{1}{Z_n^\theta(t)} \int_{-\infty < s_0 < \cdots < s_{n-1} < t} \mathbf{1}(s_0 \ge u) e^{(\theta-\lambda)u} \\
&\qquad \times \exp\big[-B(s_0) + \lambda s_0 + B_1(s_0, s_1) + \cdots + B_n(s_{n-1}, t)\big]\, ds_{0,n-1} \\
&\le \frac{Z_n^\lambda(t)}{Z_n^\theta(t)} e^{(\theta-\lambda)u}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{P}(Q_{n,t}^\theta(\sigma_0^+ \ge u) \ge e^{-su^2/n}) \;&\le\; \mathbb{P}\Big(\frac{Z_n^\lambda(t)}{Z_n^\theta(t)} e^{(\theta-\lambda)u} \ge e^{-su^2/n}\Big) \\
&=\; \mathbb{P}(\log(Z_n^\lambda(t)) - \log(Z_n^\theta(t)) \ge (\lambda - \theta)u - su^2/n).
\end{aligned}$$

From (3.3) and the distribution $e^{-r_k(t)} \sim \mathrm{Gamma}(\theta,1)$ from Theorem 3.3

$$\mathbb{E}\log(Z_n^\theta(t)) = -n\Psi_0(\theta) + \theta t. \tag{4.3}$$

Recall that $\overline{X} = X - \mathbb{E}X$ denotes a centered random variable. Consequently from above

$$\begin{aligned}
&\mathbb{P}(Q_{n,t}^\theta(\sigma_0^+ \ge u) \ge e^{-su^2/n}) \\
&\le \mathbb{P}\big\{\overline{\log(Z_n^\lambda(t))} - \overline{\log(Z_n^\theta(t))} \ge n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda - \theta) + (\lambda - \theta)u - su^2/n\big\}.
\end{aligned}$$

Given K we can restrict $b = b(K, \theta)$ small enough so that $|\lambda - \theta| \le bK \le \theta/2$ and then a Taylor expansion gives

$$|\Psi_0(\lambda) - \Psi_0(\theta) - (\lambda - \theta)\Psi_1(\theta)| \le C(\theta)(\lambda - \theta)^2.$$

Together with $|t - n\Psi_1(\theta)| \le An^{2/3}$ this leads to

$$\begin{aligned}
&n(\Psi_0(\lambda) - \Psi_0(\theta) - \frac{t}{n}(\lambda - \theta)) + (\lambda - \theta)u - su^2/n \\
&\quad\ge -C(\theta)b^2u^2/n - Abun^{-1/3} + bu^2/n - su^2/n \\
&\quad\ge \frac{bu^2}{3n}\big(1 - 3C(\theta)b\big) + \frac{bu}{3n^{1/3}}(un^{-2/3} - 3A) + \frac{bu^2}{3n}(1 - 3s/b) \\
&\quad\ge C(\theta)u^2/n.
\end{aligned}$$

The last line, with $C(\theta) > 0$, follows by enforcing $u \ge 3An^{2/3}$, choosing $s = b/3$ and then taking $b = b(\theta)$ small enough. Put this back above and apply Chebyshev's inequality:

$$\begin{aligned}
\mathbb{P}(Q^\theta_{n,t}(\sigma_0^+ \ge u) \ge e^{-su^2/n}) &\le \mathbb{P}(\overline{\log(Z_n^\lambda(t))} - \overline{\log(Z_n^\theta(t))} \ge Cu^2/n) \\
&\le C\frac{n^2}{u^4}\mathbb{V}\mathrm{ar}\big[\log Z_n^\lambda(t) - \log Z_n^\theta(t)\big] \\
&\le 2C\frac{n^2}{u^4}\big(\mathbb{V}\mathrm{ar}[\log(Z_n^\lambda(t))] + \mathbb{V}\mathrm{ar}[\log(Z_n^\theta(t))]\big). \qquad (4.4)
\end{aligned}$$

Using Lemma 4.3 below together with the definition of λ:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(\log(Z_n^\lambda(t)) &\le \mathbb{V}\mathrm{ar}(\log(Z_n^\theta(t)) + n|\Psi_1(\lambda) - \Psi_1(\theta)| \\
&\le \mathbb{V}\mathrm{ar}(\log(Z_n^\theta(t)) + C(\theta)bu \le \mathbb{V}\mathrm{ar}(\log(Z_n^\theta(t)) + C(\theta)u.
\end{aligned}$$

In the last inequality above we used the restriction $b \le 1$ which was placed on b at the outset.

Continuing from (4.4),

$$\mathbb{P}(Q^\theta_{n,t}(\sigma_0^+ \ge u) \ge e^{-su^2/n}) \quad \le \quad C\frac{n^2}{u^4}\mathbb{V}\mathrm{ar}(\log(Z_n^\theta(t)) + C\frac{n^2}{u^3}.$$

Using Theorem 3.6 and once more $u \ge 3An^{2/3}$ we get

$$\begin{aligned}
\mathbb{P}(Q^\theta_{n,t}(\sigma_0^+ \ge u) \ge e^{-su^2/n}) &\le C\frac{n^2}{u^4}E_{n,t}(\sigma_0^+) + CAn^{8/3}u^{-4} + C\frac{n^2}{u^3} \\
&\le C\frac{n^2}{u^4}E_{n,t}(\sigma_0^+) + C\frac{n^2}{u^3}
\end{aligned}$$

which is exactly what we wanted to prove.

To proof for σ^- starts with $\lambda = \theta - \frac{bu}{n}$ with $b > 0$ small. Using

$$\mathbf{1}(s_0 \le -u)e^{\theta s_0} \le e^{-(\theta-\lambda)u}\mathbf{1}(s_0 \le -u)e^{\lambda s_0}$$

we get

$$Q^\theta_{n,t}(\sigma_0^- \ge u) \le \frac{Z_n^\lambda(t)}{Z_n^\theta(t)}e^{-(\theta-\lambda)u}$$

and the rest of the proof goes the same way. □

Lemma 4.3. *For $\theta, \lambda > 0$,*

$$\big|\mathbb{V}\mathrm{ar}(\log(Z_n^\lambda(t)) - \mathbb{V}\mathrm{ar}(\log(Z_n^\theta(t))\big| \le n|\Psi_1(\lambda) - \Psi_1(\theta)|.$$

Proof: Assume $\lambda > \theta$. From Theorem 3.6

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(\log(Z_n^\lambda(t)) - \mathbb{V}\mathrm{ar}(\log(Z_n^\theta(t)) &= n(\Psi_1(\lambda) - \Psi_1(\theta)) + 2\big[E^\lambda_{n,t}(\sigma_0^+) - E^\theta_{n,t}(\sigma_0^+)\big] \\
&= -n(\Psi_1(\lambda) - \Psi_1(\theta)) + 2\big[E^\lambda_{n,t}(\sigma_0^-) - E^\theta_{n,t}(\sigma_0^-)\big].
\end{aligned}$$

From the definition (2.7) of the quenched expectation

$$\frac{\partial}{\partial\theta} E^{Q^\theta}(\sigma_0^\pm) = \mathrm{Cov}^{Q^\theta}(\sigma_0, \sigma_0^\pm).$$

For any random variable

$$\begin{aligned}\mathrm{Cov}(X, X^\pm) &= E((X^+ - X^-)X^\pm) - E(X^+ - X^-)EX^\pm \\ &= \pm\,\mathrm{Var}(X^\pm) \pm EX^+EX^-.\end{aligned}$$

Consequently

$$E^\lambda_{n,t}(\sigma_0^-) - E^\theta_{n,t}(\sigma_0^-) \le 0 \le E^\lambda_{n,t}(\sigma_0^+) - E^\theta_{n,t}(\sigma_0^+).$$

The claim follows. □

Next the tail bound for $\sigma_0^\pm$ for larger deviations.

Lemma 4.4. *Fix $0 < A < \infty$. Let $\delta > 0$. Then there exist $c = c(\theta, \delta) < \infty$ and $s = s(\theta, \delta) > 0$ such that, for all $t > 0$ and $n \ge 1$ that satisfy (2.9), and all $u \ge \max\{\delta n, 3An^{2/3}\}$,*

$$\mathbb{P}(Q_{n,t}(\sigma_0^\pm \ge u) \ge e^{-su}) \le 2e^{-cu}. \tag{4.5}$$

Proof: We do the case σ_0^+. The argument for σ_0^- is analogous. Set $\lambda = \theta + \nu$ where $\nu > 0$ is small. Then

$$\begin{aligned}\mathbb{P}(Q^\theta_{n,t}(\sigma_0^+ \ge u) \ge e^{-su}) &\le \mathbb{P}\Big(\frac{Z^\lambda_n(t)}{Z^\theta_n(t)} e^{(\theta-\lambda)u} \ge e^{-su}\Big) \\ &= \mathbb{P}(\log(Z^\lambda_n(t)) - \log(Z^\theta_n(t)) \ge (\lambda - \theta)u - su).\end{aligned}$$

After centering:

$$\begin{aligned}&\mathbb{P}(Q^\theta_{n,t}(\sigma_0^+ \ge u) \ge e^{-su}) \\ &\quad\le \mathbb{P}\big(\overline{\log(Z^\lambda_n(t))} - \overline{\log(Z^\theta_n(t))} \ge n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda - \theta) + (\lambda - \theta)u - su\big).\end{aligned}$$

From a Taylor expansion, $u \ge \max\{\delta n, 4An^{2/3}\}$, and then fixing ν small enough and $s = \nu/2$,

$$\begin{aligned}&n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda - \theta) + (\lambda - \theta)u - su \\ &\quad\ge -C(\theta)\nu^2 n - A\nu n^{2/3} + \nu u - su \\ &\quad\ge C(\delta, \theta)u.\end{aligned}$$

From (3.7) and (3.5)

$$\overline{\log(Z^\lambda_n(t))} - \overline{\log(Z^\theta_n(t))} = \sum_{j=1}^n \overline{r^\lambda_j(t)} - \sum_{j=1}^n \overline{r^\theta_j(t)} \tag{4.6}$$

where, for a fixed t, $e^{-r^\theta_j(t)}$ are i.i.d. Gamma$(\theta, 1)$ variables, and similarly $e^{-r^\lambda_j(t)}$with parameter λ. Thus for certain sums S'_n, S''_n of i.i.d. mean zero variables with an exponential moment

$$\begin{aligned}\mathbb{P}(Q^\theta_{n,t}(\sigma_0^+ \ge u) \ge e^{-su}) &\le \mathbb{P}(S'_n - S''_n \ge Cu) \\ &\le \mathbb{P}(S'_n \ge Cu/2) + \mathbb{P}(S''_n < -Cu/2) \le e^{-cu}.\end{aligned}$$

The last inequality follows from $u \ge \delta n$ and standard large deviation theory. □

Now we combine the two deviation estimates into a moment bound.

Lemma 4.5. *Fix $0 < A < \infty$. Then there exists $C(\theta) < \infty$ such that, for all $t > 0$ and $n \geq 1$ that satisfy (2.9),*

$$E_{n,t}(\sigma_0^{\pm}) \leq (4A + C(\theta))n^{2/3}. \tag{4.7}$$

Proof: We do the computation for σ_0^+. It is identical for σ_0^-. Let $r \geq 1$, to be chosen at the end, and $B = r \vee (3A)$. Begin with

$$E_{n,t}(\sigma_0^+) \leq Bn^{2/3} + \int_{Bn^{2/3}}^{n\vee(Bn^{2/3})} P(\sigma_0^+ \geq u)\,du + \int_{n\vee(Bn^{2/3})}^{\infty} P(\sigma_0^+ \geq u)\,du. \tag{4.8}$$

The last integral in (4.8) is bounded by a constant that depends on θ, uniformly over $r > 0$, $A > 0$ and $n \geq 1$, as can be seen by an application of Lemma 4.4 with $\delta = 1$. The middle integral is bounded as follows, utilizing Lemma 4.2 with $K = 1$.

$$\begin{aligned}
&\int_{Bn^{2/3}}^{n\vee(Bn^{2/3})} P(\sigma_0^+ \geq u)\,du \\
&\quad\leq \int_{Bn^{2/3}}^{n\vee(Bn^{2/3})} \Big\{ e^{-su^2/n} + \mathbb{P}(Q_{n,t}(\sigma_0^+ \geq u) \geq e^{-su^2/n}) \Big\}\,du \\
&\quad\leq \int_{Bn^{2/3}}^{\infty} \left(\frac{C(\theta)n^2}{u^4} E_{n,t}(\sigma_0^+) + \frac{C(\theta)n^2}{u^3} \right) du + C(\theta) \\
&\quad\leq \frac{C(\theta)}{B^3} E_{n,t}(\sigma_0^+) + \frac{C(\theta)}{B^2} n^{2/3} + C(\theta).
\end{aligned}$$

Combine the estimates, noting that $B \geq 1$ and $n \geq 1$, to

$$E_{n,t}(\sigma_0^+) \leq (B + C(\theta))n^{2/3} + \frac{C(\theta)}{r^3} E_{n,t}(\sigma_0^+).$$

Choose $r = (4C(\theta))^{1/3}$. Rearranging gives the conclusion. □

Theorem 4.1 is now proved by (4.7), the variance identity (3.19) and assumption (2.9). For future reference let us also state tail bounds on $\sigma_0^{\pm}$ that we obtain by combining (4.7) with Lemmas 4.2 and 4.4.

Proposition 4.6. *Under assumption (2.9) we have these tail bounds, for finite positive constants C, c and s that depend on θ, and for $b > 0$:*

$$\mathbb{P}(Q_{n,t}({\sigma_0}^{\pm} \geq bn^{2/3}) \geq e^{-sb^2n^{1/3}}) \leq Cb^{-3} \quad \textit{for } 3A \leq b \leq n^{1/3}, \tag{4.9}$$

$$\mathbb{P}(Q_{n,t}({\sigma_0}^{\pm} \geq bn^{2/3}) \geq e^{-sbn^{2/3}}) \leq 2e^{-cbn^{2/3}} \quad \textit{for } b \geq n^{1/3} \vee (3A) \tag{4.10}$$

and

$$P_{n,t}(\sigma_0^{\pm} \geq bn^{2/3}) \leq Cb^{-3} \quad \textit{for } b \geq 3A. \tag{4.11}$$

From (4.11) we get the moment bound

$$E_{n,t}(\,|\sigma_0|^p\,) \leq (3^pA^p + C(\theta))n^{2p/3} \quad \text{for } 1 \leq p < 3. \tag{4.12}$$

5. Lower bound on the variance

Theorem 5.1. *Fix $0 < A < \infty$. Then there exists a constant $C_1 = C_1(A,\theta)$ such that, for $t > 0$ and $n \ge 1$ that satisfy (2.9),*

$$\mathbb{V}\text{ar}(\log Z_n^\theta(t)) \ge C_1 n^{2/3}. \tag{5.1}$$

The estimate that gives the theorem is in the next proposition.

Proposition 5.2. *Assume that (2.9) holds with a constant $A \in \mathbb{R}_+$. Then there exist finite positive θ-dependent constants $C(\theta), c(\theta), D(\theta)$ so that, if $0 < \delta \le 1$ and $K \ge 1$ satisfy*

$$D(\theta)(A+1)\delta^{1/2} \le K \le c(\theta)(A+1)^{-4}\delta^{-1/2},$$

then

$$\varlimsup_{n\to\infty} \mathbb{P}\big(Q^\theta_{n,t}(0 < \sigma_0 \le \delta n^{2/3}) > e^{-Kn^{1/3}\sqrt{\delta}}\big) \le C(\theta)(e^{-K^2/16} + K^{3/4}\delta^{3/8}).$$

Remark 5.3. As a corollary we get the following more general statement. Fix $x \in \mathbb{R}$ and assume that

$$D(\theta)(A+|x|+1)\delta^{1/2} \le K \le c(\theta)(A+|x|+1)^{-4}\delta^{-1/2}. \tag{5.2}$$

Then

$$\begin{aligned}\varlimsup_{n\to\infty} \mathbb{P}(Q^\theta_{n,t}(xn^{2/3} < \sigma_0 \le (x+\delta)n^{2/3}) > e^{-Kn^{1/3}\sqrt{\delta}})\\ \le C(e^{-K^2/16} + K^{3/4}\delta^{3/8}).\end{aligned} \tag{5.3}$$

This follows because by the translation invariance (3.10)

$$Q^\theta_{n,t}(xn^{2/3} < \sigma_0 \le (\delta+x)n^{2/3}) \overset{d}{=} Q^\theta_{n,t-xn^{2/3}}(0 < \sigma_0 \le \delta n^{2/3})$$

and $|t - xn^{2/3} - n\Psi_1(\theta)| \le (A+|x|)n^{2/3}$. In particular, with $x = -\delta$, we get the matching estimate for σ_0^-, and we can combine the estimates for $\sigma_0^\pm$: under assumptions (2.9) and (5.2) with $x = -\delta$,

$$\varlimsup_{n\to\infty} \mathbb{P}\big(Q^\theta_{n,t}(|\sigma_0| \le \delta n^{2/3}) > 2e^{-Kn^{1/3}\sqrt{\delta}}\big) \le C(\theta)(e^{-K^2/16} + K^{3/4}\delta^{3/8}). \tag{5.4}$$

Before proving Proposition 5.2 let us observe how Theorem 5.1 is proved. Estimate (5.4) gives the annealed limit

$$\lim_{\delta\searrow 0}\varlimsup_{n\to\infty} P^\theta_{n,t}(|\sigma_0| \le \delta n^{2/3}) = 0.$$

Then by the variance identity (3.19)

$$\mathbb{V}\text{ar}(\log Z_n^\theta(t)) = E^\theta_{n,t}\big(|\sigma_0|\big) \ge \delta n^{2/3} P^\theta_{n,t}(|\sigma_0| \ge \delta n^{2/3})$$

and Theorem 5.1 follows.

Proof of Proposition 5.2: Set $u = \delta n^{2/3}, \upsilon(\delta) = K\sqrt{\delta}$ and begin by writing

$$\begin{aligned}
&\mathbb{P}(Q^\theta_{n,t}(0 < \sigma_0 \le \delta n^{2/3}) > e^{-n^{1/3}\upsilon(\delta)}) \\
&\qquad = \mathbb{P}\left(\frac{Q^\theta(\sigma_0 > u \text{ or } \sigma_0 < 0)}{Q^\theta(0 < \sigma_0 \le u)} < e^{n^{1/3}\upsilon(\delta)} - 1\right) \\
&\qquad = \mathbb{P}\left(\frac{Z^\theta_{n,t}(\sigma_0 > u \text{ or } \sigma_0 < 0)}{Z^\theta_{n,t}(0 < \sigma_0 \le u)} < e^{n^{1/3}\upsilon(\delta)} - 1\right) \\
&\qquad = \mathbb{P}\left(\frac{Z^\theta_{n,t}(\sigma_0 > u \text{ or } \sigma_0 < 0)}{Z^\theta_{n,t}(0 < \sigma_0 \le u)} \cdot \frac{Z_{1,n}(0,t)}{Z_{1,n}(0,t)} < e^{n^{1/3}\upsilon(\delta)} - 1\right).
\end{aligned}$$

Split the last probability to get

$$\begin{aligned}
&\mathbb{P}(Q^\theta_{n,t}(0 < \sigma_0 \le \delta n^{2/3}) > e^{-n^{1/3}\upsilon(\delta)}) \\
&\qquad \le \ \mathbb{P}\left(\frac{Z^\theta_{n,t}(\sigma_0 > u \text{ or } \sigma_0 < 0)}{Z_{1,n}(0,t)} < e^{2n^{1/3}\upsilon(\delta)}\right) \\
&\qquad\qquad + \mathbb{P}\left(\frac{Z^\theta_{n,t}(0 < \sigma_0 \le u)}{Z_{1,n}(0,t)} > e^{n^{1/3}\upsilon(\delta)} \frac{1}{1 - e^{-n^{1/3}\upsilon(\delta)}}\right) \\
&\qquad \le \ \mathbb{P}\left(\frac{Z^\theta_{n,t}(\sigma_0 > u)}{Z_{1,n}(0,t)} < e^{2n^{1/3}\upsilon(\delta)}\right) && (5.5) \\
&\qquad\qquad + \ \mathbb{P}\left(\frac{Z^\theta_{n,t}(0 < \sigma_0 \le u)}{Z_{1,n}(0,t)} > e^{n^{1/3}\upsilon(\delta)}\right). && (5.6)
\end{aligned}$$

We bound probabilities in (5.5) and (5.6) separately.

The term (5.5).

$$\frac{Z^\theta_{n,t}(\sigma_0 > u)}{Z_{1,n}(0,t)} = \int_u^t \exp(-B(s) + \theta s) \frac{Z_{1,n}(s,t)}{Z_{1,n}(0,t)} ds.$$

Construct a new environment $\tilde{\omega}$ with

$$\tilde{B}(s) = -(B_n(t) - B_n(t-s)), \quad \tilde{B}_i(s) = B_{n-i}(t) - B_{n-i}(t-s), \quad 1 \le i \le n-1,$$

and take a new parameter $\lambda = \theta + a(\delta)n^{-1/3}$ where $a(\delta) = K^{-1/4}\delta^{-1/8}$. Quantities that use environment $\tilde{\omega}$ are marked with a tilde. From the definitions one checks that

$$Z_{1,n}(s,t) = \tilde{Z}_{0,n-1}(0, t-s) \quad \text{for any } t > 0 \text{ and } s \in (-\infty, t). \tag{5.7}$$

Use (3.30) for the new system to get

$$\begin{aligned}\frac{Z_{1,n}(s,t)}{Z_{1,n}(0,t)} = \frac{\tilde{Z}_{0,n-1}(0,t-s)}{\tilde{Z}_{0,n-1}(0,t)} &\geq \frac{\tilde{Z}^{\lambda}_{n-1,t-s}(\sigma_0 > 0)}{\tilde{Z}^{\lambda}_{n-1,t}(\sigma_0 > 0)} \\ &= \frac{\tilde{Q}^{\lambda}_{n-1,t-s}(\sigma_0 > 0)\tilde{Z}^{\lambda}_{n-1,t-s}}{\tilde{Q}^{\lambda}_{n-1,t}(\sigma_0 > 0)\tilde{Z}^{\lambda}_{n-1,t}} \\ &\geq \tilde{Q}^{\lambda}_{n-1,t-s}(\sigma_0 > 0)\frac{\tilde{Z}^{\lambda}_{n-1,t-s}}{\tilde{Z}^{\lambda}_{n-1,t}} \\ &= \tilde{Q}^{\lambda}_{n-1,t-s}(\sigma_0 > 0)\exp(\tilde{Y}_{n-1}(t-s,t) - \lambda s).\end{aligned}$$

Thus, denoting the probability in (5.5) by p_1,

$$\begin{aligned}p_1 &= \mathbb{P}\left(\frac{Z^{\theta}_{n,t}(\sigma_0 > u)}{Z_{1,n}(0,t)} < e^{2n^{1/3}\upsilon(\delta)}\right) \\ &\leq \mathbb{P}\left(\int_u^t e^{-B(s)+\tilde{Y}_{n-1}(t-s,t)+(\theta-\lambda)s}\,\tilde{Q}^{\lambda}_{n-1,t-s}(\sigma_0 > 0)\,ds < e^{2n^{1/3}\upsilon(\delta)}\right) \\ &= \mathbb{P}\left(\int_u^t e^{-B(s)+\tilde{Y}^*_{n-1}(t-s,t)+(\theta-\lambda)s}\,\tilde{Q}^{\lambda,*}_{n-1}(t-s-\sigma^*_{n-1} > 0)\,ds < e^{2n^{1/3}\upsilon(\delta)}\right).\end{aligned}$$

On the last line above we applied the $*$ transformation to the $\tilde{\omega}$ system and Lemma 3.5 to replace the measure $\tilde{Q}^{\lambda,\tilde{\omega}^*}$ with the dual measure $\tilde{Q}^{\lambda,*}$.

Set

$$\bar{u} = n\Psi_1(\theta) - (n-1)\Psi_1(\lambda).$$

Given any δ and K, there exists a constant $c_0(\theta) > 0$ such that $\bar{u}/2 \geq c_0(\theta)a(\delta)n^{2/3}$ for large enough n. Furthermore, from the hypothesis on δ and K it follows that $\bar{u}/2 \geq u$ and $\bar{u}/2 \geq 3An^{2/3}$. Restrict the integration inside the probability and decompose again:

$$\begin{aligned}p_1 &\leq \mathbb{P}\left(\int_u^{\bar{u}/2} e^{-B(s)+\tilde{Y}^*_{n-1}(t-s,t)+(\theta-\lambda)s}\,\tilde{Q}^{\lambda,*}_{n-1}(t-\sigma^*_{n-1} > \bar{u}/2)\,ds < e^{2n^{1/3}\upsilon(\delta)}\right) \\ &\leq \mathbb{P}\left(\tilde{Q}^{\lambda,*}_{n-1}(t-\sigma^*_{n-1} > \bar{u}/2) \leq 1/2\right) \qquad (5.8) \\ &\quad + \mathbb{P}\left(\int_u^{\bar{u}/2} \exp(-B(s) + \tilde{Y}^*_{n-1}(t-s,t) + (\theta-\lambda)s)ds < 2e^{2n^{1/3}\upsilon(\delta)}\right). \qquad (5.9)\end{aligned}$$

For probability (5.8) switch to complements, apply Lemma 3.5 again, and then the upper bound (4.9):

$$\begin{aligned}&\mathbb{P}\left(\tilde{Q}^{\lambda,*}_{n-1}(t-\sigma^*_{n-1} > \bar{u}/2) \leq 1/2\right) \\ &\qquad = \mathbb{P}\left(\tilde{Q}^{\lambda,*}_{n-1}(t-\bar{u}-\sigma^*_{n-1} \leq -\bar{u}/2) > 1/2\right) \\ &\qquad = \mathbb{P}\left(\tilde{Q}^{\lambda}_{n-1,t-\bar{u}}(\sigma_0 \leq -\bar{u}/2) > 1/2\right) \\ &\qquad \leq C(\theta)a(\delta)^{-3} = C(\theta)K^{3/4}\delta^{3/8}.\end{aligned} \tag{5.10}$$

On the last line above assumption (2.9) continues to be valid with the same constant A because $|t - \bar{u} - (n-1)\Psi_1(\lambda)| = |t - n\Psi_1(\theta)| \leq An^{2/3}$. Property $\bar{u}/2 \geq 3An^{2/3}$ is the assumption needed for (4.9).

Next we estimate probability (5.9). Process $s \mapsto \tilde{Y}^*_{n-1}(t-s,t)$ is a standard Brownian motion, and independent of B because $\tilde{Y}^*$ was constructed from the new environment $\tilde{\omega}$. Define another standard Brownian motion

$$B^\dagger(s) = 2^{-1/2}n^{-1/3}\big(-B(n^{2/3}s) + \tilde{Y}^*_{n-1}(t-n^{2/3}s,t)\big).$$

Then probability (5.9) equals

$$\begin{aligned}
&\mathbb{P}\left(\int_u^{\bar{u}/2} \exp(\sqrt{2}n^{1/3}B^\dagger(n^{-2/3}s) + (\theta-\lambda)s)\,ds < 2e^{2n^{1/3}\upsilon(\delta)}\right)\\
&\quad\le \mathbb{P}\left(n^{2/3}\int_\delta^{c_0(\theta)a(\delta)} \exp(\sqrt{2}n^{1/3}B^\dagger(s) - a(\delta)n^{1/3}s)\,ds < 2e^{2n^{1/3}\upsilon(\delta)}\right)\\
&\quad= \mathbb{P}\left(n^{-1/3}\log\int_\delta^{c_0(\theta)a(\delta)} e^{\sqrt{2}n^{1/3}B^\dagger(s)-a(\delta)n^{1/3}s}\,ds \le 2\upsilon(\delta) + \frac{\log(2n^{-2/3})}{n^{1/3}}\right)
\end{aligned}$$

As $n\to\infty$ the probability on the last line above converges to

$$\mathbb{P}\left(\sup_{\delta\le s\le c_0(\theta)a(\delta)} (\sqrt{2}B^\dagger(s) - a(\delta)s) \le 2\upsilon(\delta)\right).$$

Introduce one more Brownian motion $B(s) = B^\dagger(\delta+s) - B^\dagger(\delta)$. Abbreviate temporarily $\tau = a(\delta)^2(c_0(\theta)a(\delta)-\delta)/\sqrt{2} > 1$ where the inequality is a consequence of the assumption on δ and K. Then the probability above is

$$\begin{aligned}
&\le \mathbb{P}\big(\sqrt{2}B^\dagger(\delta) < -\upsilon(\delta)\big) + \mathbb{P}\left(\sup_{0\le s\le c_0(\theta)a(\delta)-\delta} (\sqrt{2}B(s) - a(\delta)s) \le 3\upsilon(\delta) + \delta a(\delta)\right)\\
&\le e^{-\frac{1}{4}\upsilon(\delta)^2\delta^{-1}} + \mathbb{P}\left(\sup_{0\le s\le\tau}(B(s)-s) \le 2^{-1/2}a(\delta)(3\upsilon(\delta)+\delta a(\delta))\right)\\
&\le e^{-\frac{1}{4}\upsilon(\delta)^2\delta^{-1}} + \mathbb{P}\left(\sup_{0\le s\le 1}(B(s)-s) \le 2^{-1/2}a(\delta)(3\upsilon(\delta)+\delta a(\delta))\right)\\
&\le e^{-\frac{1}{4}\upsilon(\delta)^2\delta^{-1}} + Ca(\delta)(3\upsilon(\delta)+\delta a(\delta)).
\end{aligned}$$

The last inequality comes because $\sup_{0\le s\le 1}(B(s)-s)$ is a.s. positive with a bounded density function. Including the estimate from (5.10) we get

$$\begin{aligned}
\varlimsup_{n\to\infty}\,(5.5) &\le C(\theta)(e^{-K^2/4} + K^{3/4}\delta^{3/8} + K^{-1/2}\delta^{3/4})\\
&\le C(\theta)(e^{-K^2/4} + K^{3/4}\delta^{3/8}).
\end{aligned} \tag{5.11}$$

In the last inequality we used $\delta \le 1 \le K$.

The term (5.6).

For probability (5.6) we separate the argument into a lemma because the same estimate will be needed again, though with different parameters.

Lemma 5.4. *Assume (2.9) with the constant A and let $a, b, \kappa > 0$. Then there exist finite, positive constants $C(\theta)$, $C_1(\theta)$ and $n_0(a,b,\kappa,\theta)$ such that, if $b/a \ge C_1(\theta)(A+1)$, then for $n \ge n_0(a,b,\kappa,\theta)$*

$$\mathbb{P}\left(\frac{Z^\theta_{n,t}(0<\sigma_0^\pm \le an^{2/3})}{Z_{1,n}(0,t)} \ge \kappa e^{n^{1/3}b}\right) \le C(\theta)\big(a^3b^{-3} + \exp(-b^2a^{-1}/16)\big).$$

Before proving the lemma let us use it to conclude the proof of Proposition 5.2. In Lemma 5.4 take $\kappa = 1$, $a = \delta$ and $b = K\sqrt{\delta}$. Then for large enough n,

$$\text{probability (5.6)} \ \le C(\theta)(\delta^{3/2}K^{-3} + e^{-K^2/16}). \tag{5.12}$$

Combine (5.11) and (5.12) with $\delta \le 1 \le K$, and we have

$$\varlimsup_{n\to\infty} \mathbb{P}(Q^{\theta}_{n,t}(0 < \sigma_0 \le \delta n^{2/3}) > e^{-n^{1/3}\upsilon(\delta)}) \le C(\theta)(e^{-K^2/16} + K^{3/4}\delta^{3/8})$$

and the proposition is proved. $\square$

Proof of Lemma 5.4: We do the case of σ_0^+ in full detail. Abbreviate $u = an^{2/3}$. Introduce the new environment $\tilde{\omega}$ as before, and a new parameter $\lambda = \theta - rn^{-1/3}$ with $r = b/(4a)$. We must restrict n large enough so that for example $rn^{-1/3} < \theta/2$ so that λ is a legitimate parameter.

Begin with (5.7) and then apply comparison (3.30):

$$\begin{aligned}
\frac{Z_{1,n}(s,t)}{Z_{1,n}(0,t)} &= \frac{\tilde{Z}_{0,n-1}(0,t-s)}{\tilde{Z}_{0,n-1}(0,t)} \le \frac{\tilde{Z}^{\lambda}_{n-1,t-s}(\sigma_0 < 0)}{\tilde{Z}^{\lambda}_{n-1,t}(\sigma_0 < 0)} = \frac{\tilde{Z}^{\lambda}_{n-1,t-s}}{\tilde{Z}^{\lambda}_{n-1,t}} \cdot \frac{\tilde{Q}^{\lambda}_{n-1,t-s}(\sigma_0 < 0)}{\tilde{Q}^{\lambda}_{n-1,t}(\sigma_0 < 0)} \\
&\le \exp\bigl(\tilde{Y}_{n-1}(t-s,t) - \lambda s\bigr) \cdot \frac{1}{\tilde{Q}^{\lambda}_{n-1,t}(\sigma_0 < 0)}.
\end{aligned}$$

Substitute the above bound in the probability that is to be bounded:

$$\begin{aligned}
&\mathbb{P}\left(\frac{Z^{\theta}_{n,t}(0 < \sigma_0 \le u)}{Z_{1,n}(0,t)} \ge \kappa e^{n^{1/3}b}\right) \\
&\quad = \mathbb{P}\left(\int_0^u \exp(-B(s) + \theta s)\frac{Z_{1,n}(s,t)}{Z_{1,n}(0,t)}\,ds \ge \kappa e^{n^{1/3}b}\right) \\
&\quad \le \mathbb{P}\left(\int_0^u \frac{\exp(-B(s) + \tilde{Y}_{n-1}(t-s,t) + (\theta-\lambda)s)}{\tilde{Q}^{\lambda}_{n-1,t}(\sigma_0 < 0)}\,ds \ge \kappa e^{n^{1/3}b}\right) \\
&\quad \le \mathbb{P}\left(\tilde{Q}^{\lambda}_{n-1,t}(\sigma_0 < 0) \le 1/2\right) \qquad (5.13) \\
&\qquad + \ \mathbb{P}\left(\int_0^u \exp(-B(s) + \tilde{Y}_{n-1}(t-s,t) + (\theta-\lambda)s)\,ds \ge \frac{\kappa}{2}e^{n^{1/3}b}\right). \qquad (5.14)
\end{aligned}$$

To treat probability (5.13) set $\bar{u} = (n-1)\Psi_1(\lambda) - n\Psi_1(\theta)$. Ψ_1 is positive, convex and strictly decreasing, so one can check that $\bar{u} \ge \frac{1}{4}|\Psi_1'(\theta)|rn^{2/3}$ for all $n \ge 1$ provided $C_1(\theta)$ in the hypothesis is large enough. Use the shift invariance property of Q described in Remark 3.1 and the upper bound (4.9):

$$\begin{aligned}
&\mathbb{P}(\tilde{Q}^{\lambda}_{n-1,t}(\sigma_0 < 0) < 1/2) = \mathbb{P}(\tilde{Q}^{\lambda}_{n-1,t}(\sigma_0 > 0) \ge 1/2) \\
&\qquad = \mathbb{P}(\tilde{Q}^{\lambda}_{n-1,t+\bar{u}}(\sigma_0 > \bar{u}) \ge 1/2) \le C(\theta)r^{-3} \le C(\theta)(a/b)^3.
\end{aligned} \tag{5.15}$$

The choice of $\bar{u}$ makes (2.9) valid again with the same A, and a large enough $C_1(\theta)$ guarantees that $\bar{u} \ge 3An^{2/3}$ so that (4.9) can be applied.

For probability (5.14), after rescaling the integral and introducing a new Brownian motion,

$$
\begin{aligned}
(5.14) &\le \mathbb{P}\left(n^{2/3}\int_0^a \exp(n^{1/3}(\sqrt{2}B^\dagger(s)+rs))ds > \frac{\kappa}{2}e^{n^{1/3}b}\right)\\
&= \mathbb{P}\left(n^{-1/3}\log\int_0^a \exp(n^{1/3}(\sqrt{2}B^\dagger(s)+rs))ds > b + n^{-1/3}\log(\kappa n^{-2/3}/2)\right)\\
&\le \mathbb{P}\left(\sup_{0\le s\le a}(\sqrt{2}B^\dagger(s)+rs) \ge \tfrac{3}{4}b\right).
\end{aligned}
$$

In the last inequality we took n large enough so that $n^{-1/3}\log(\kappa n^{-2/3}/2) < b/4$. Via $\sup_{0\le s\le a} B^\dagger(s) \overset{d}{=} a^{1/2}|B^\dagger(1)|$ bound the last probability by

$$
\mathbb{P}\left(\sup_{0\le s\le a}\sqrt{2}B^\dagger(s) \ge \tfrac{3}{4}b - ra\right) \le C\exp\left(-\frac{1}{4a}(\tfrac{3}{4}b-ra)^2\right) = C\exp\left(-\frac{b^2}{16a}\right).
$$

Combining estimate (5.15) with above gives the conclusion for σ_0^+.

The case of σ_0^- goes similarly, with small alterations. Now $\lambda = \theta + rn^{-1/3}$. Utilizing (5.7) and comparison (3.30) the ratio is developed as follows:

$$
\begin{aligned}
\frac{Z^\theta_{n,t}(-u\le\sigma_0<0)}{Z_{1,n}(0,t)} &= \int_{-u}^0 \exp(-B(s)+\theta s)\frac{Z_{1,n}(s,t)}{Z_{1,n}(0,t)}\,ds\\
&\le \int_{-u}^0 \frac{\exp(-B(s)-\tilde{Y}_{n-1}(t,t-s)-(\theta-\lambda)s)}{\tilde{Q}^\lambda_{n-1,t}(\sigma_0>0)}\,ds.
\end{aligned}
$$

The rest follows along the same lines as above. With this we consider Lemma 5.4 proved. □

6. Fluctuations of the path under boundary conditions

Theorem 6.1. *Assume $t>0$ and $n\ge 1$ satisfy (2.9) for a fixed $A<\infty$, let $0<\gamma<1$ and assume $b\ge 3(A+1)$. Then for $n\ge(1-\gamma)^{-1}$*

$$
P(|\sigma_{\lfloor\gamma n\rfloor}-\gamma t| > bn^{2/3}) \le C(\theta)b^{-3}. \tag{6.1}
$$

Also, for any $0<\gamma<1, \varepsilon>0$ there exists $\delta>0$ with

$$
\varlimsup_{n\to\infty} P(|\sigma_{\lfloor\gamma n\rfloor}-\gamma t| \le \delta n^{2/3}) \le \varepsilon. \tag{6.2}
$$

Proof: For the first statement it is enough to prove that

$$
Q_{n,t}(\sigma_k - v > u) \overset{d}{=} Q_{n-k,t-v}(\sigma_0>u). \tag{6.3}
$$

Indeed, from this identity we get

$$
P_{n,t}(|\sigma_{\lfloor\gamma n\rfloor}-\gamma t|>bn^{2/3}) = P_{n-\lfloor\gamma n\rfloor,(1-\gamma)t}(|\sigma_0|>bn^{2/3}) \le C(\theta)b^{-3} \tag{6.4}
$$

where the last inequality comes from applying (4.11). This is legitimate because

$$
|(1-\gamma)t-(n-\lfloor\gamma n\rfloor)\Psi_1(\theta)| \le (A+1)n^{2/3}.
$$

Condition $n\ge(1-\gamma)^{-1}$ ensures that $n-\lfloor\gamma n\rfloor\ge 1$.

By Lemma 3.5, to prove (6.3) it is enough to show that the distribution of $(\sigma_1^*,\dots,\sigma_{n-1}^*)$ is the same under Q_n^* and Q_{n-1}^*. For this we check that integrating out σ_n^* from the density function of Q_n^* results in the density of Q_{n-1}^*. In the next calculation use (3.16) and (3.2).

$$\begin{aligned}
&\int_{s_{n-1}}^{\infty} \frac{1}{Z_n^*} \exp\left[X_1(0,s_1) + X_2(s_1,s_2) + \cdots + X_n(s_{n-1},s_n) + Y_n(0,s_n) - \theta s_n\right] ds_n \\
&= \frac{1}{Z_n^*} \exp\left[X_1(0,s_1) + \cdots + X_{n-1}(s_{n-2},s_{n-1}) + Y_n(0,s_{n-1}) - \theta s_{n-1}\right] \\
&\qquad \times \int_{s_{n-1}}^{\infty} \exp\left[X_n(s_{n-1},s_n) + Y_n(s_{n-1},s_n) - \theta(s_n - s_{n-1})\right] ds_n \\
&= \frac{1}{Z_n^*} \exp\left[X_1(0,s_1) + \cdots + X_{n-1}(s_{n-2},s_{n-1}) + Y_n(0,s_{n-1}) - \theta s_{n-1}\right] e^{r_n(s_{n-1})} \\
&= \frac{e^{r_n(0)}}{Z_n^*} \exp\left[X_1(0,s_1) + \cdots + X_{n-1}(s_{n-2},s_{n-1}) + Y_{n-1}(0,s_{n-1}) - \theta s_{n-1}\right]
\end{aligned}$$

which is exactly the density of Q_{n-1}^* and also shows that $Z_n^* = Z_{n-1}^* e^{r_n(0)}$.

To prove (6.2), use (6.3) to write

$$Q_{n,t}(|\sigma_{\lfloor \gamma n \rfloor} - \gamma t| \le \delta n^{2/3}) \overset{d}{=} Q_{n-\lfloor \gamma n \rfloor,(1-\gamma)t}(|\sigma_0| \le \delta n^{2/3})$$

and apply Proposition 5.2. □

7. Upper bounds without boundary conditions

Theorem 7.1. *Let $\tau > 0$ and pick θ so that $\Psi_1(\theta) = \tau$. Then for $n \ge n_0(\tau)$ and $b \ge b_0(\tau)$ we have*

$$\mathbb{P}(|\log Z_{1,n}(0,n\tau) - n(\Psi_1(\theta)\theta - \Psi_0(\theta))| \ge bn^{1/3}) \le C(\tau)b^{-3/2}. \tag{7.1}$$

Proof: The choice of θ gives $\mathbb{E}(\log Z_{n,n\tau}^\theta) = n(\Psi_1(\theta)\theta - \Psi_0(\theta))$, so by Theorem 4.1 we only need to prove the bound

$$\mathbb{P}(|\log Z_{1,n}(0,n\tau) - \log Z_{n,n\tau}^\theta| \ge bn^{1/3}) \le Cb^{-3/2}. \tag{7.2}$$

Abbreviate $t = n\tau$. By (3.4)

$$Z_{n,t}^\theta \ge e^{r_1(0)} Z_{1,n}(0,t). \tag{7.3}$$

This gives

$$\mathbb{P}\left(\log Z_{n,t}^\theta - \log Z_{1,n}(0,t) \le -bn^{1/3}\right) \le \mathbb{P}\left(e^{r_1(0)} \le e^{-bn^{1/3}}\right) \le Ce^{-bn^{1/3}},$$

the last inequality follows from $e^{r_1(0)} \sim \text{Gamma}(\theta,1)^{-1}$ which has bounded density near 0.

To get the opposite bound set $u = \sqrt{b}n^{2/3}$ and write

$$\begin{aligned}
\mathbb{P}\left(\frac{Z_{n,t}^\theta}{Z_{1,n}(0,t)} \ge e^{bn^{1/3}}\right) &= \mathbb{P}\left(\frac{Z_{n,t}^\theta(|\sigma_0| \le u)}{Z_{1,n}(0,t)\, Q_{n,t}^\theta(|\sigma_0| \le u)} \ge e^{bn^{1/3}}\right) \\
&\le \mathbb{P}\left(\frac{Z_{n,t}^\theta(|\sigma_0| \le u)}{Z_{1,n}(0,t)} \ge \frac{1}{2}e^{bn^{1/3}}\right) + \mathbb{P}\left(Q_{n,t}^\theta(|\sigma_0| \le u) \le 1/2\right) \\
&\le C(\theta)b^{-3/2}.
\end{aligned} \tag{7.4}$$

To get the last inequality, apply Lemma 5.4 with $a = \sqrt{b}$ to the first probability, and the upper bound (4.9) to the second probability, and take both n and b large enough. $\square$

Theorem 1.2 is a restatement of Theorem 7.1. The next theorem proves Theorem 2.1.

Theorem 7.2. *Assume* (2.9) *holds, and* $0 < \gamma < 1$. *Then for large enough* n *and* b *we have*

$$P_{(1,n),(0,t)}\left(|\sigma_{\lfloor n\gamma\rfloor} - \gamma t| > bn^{2/3}\right) \le C(\theta)b^{-3}. \tag{7.5}$$

Proof: Let $\ell = \lfloor n\gamma\rfloor, t' = \gamma t$ and $u = bn^{2/3}$. By the definitions and (7.3)

$$\begin{aligned} Q_{(1,n),(0,t)}\left(|\sigma_\ell - t'| > u\right) &= \frac{1}{Z_{1,n}(0,t)} \int\limits_{|s-t'|>u} Z_{1,\ell}(0,s) Z_{\ell+1,n}(s,t)\,ds \\ &\le \frac{e^{-r_1(0)}}{Z_{1,n}(0,t)} \int\limits_{|s-t'|>u} Z^\theta_{\ell,s}\, Z_{\ell+1,n}(s,t)\,ds \;=\; \frac{e^{-r_1(0)} Z^\theta_{n,t}}{Z_{1,n}(0,t)} Q_{n,t}\left(|\sigma_\ell - t'| > u\right). \end{aligned}$$

Consider $h \in (b^{-3}, 1)$.

$$\begin{aligned} \mathbb{P}\left(Q_{(1,n),(0,t)}\left(|\sigma_\ell - t'| > u\right) > h\right) \;&\le\; \mathbb{P}(e^{r_1(0)} \le b^{-3}) + \mathbb{P}\left[\frac{Z^\theta_{n,t}}{Z_{1,n}(0,t)} \ge e^{rn^{1/3}}\right] \\ &\quad + \mathbb{P}\left[Q_{n,t}\left(|\sigma_\ell - t'| > u\right) > e^{-rn^{1/3}} hb^{-3}\right] \end{aligned}$$

where we set $r = sb^2/(3(1-\gamma))$ with s from Proposition 4.6. The first term is bounded by Cb^{-3} as $e^{r_1(0)}$ has bounded density near zero. The second term is bounded by $Cr^{-3/2} \le Cb^{-3}$ by (7.4). Finally, (6.3) and Lemma 4.2 give, for large enough n and b and uniformly for $h \in (b^{-3}, 1)$,

$$\begin{aligned} \mathbb{P}\left[Q_{n,t}\left(|\sigma_\ell - t'| > u\right) > e^{-rn^{1/3}} hb^{-3}\right] &\le \mathbb{P}\left[Q_{n,t}\left(|\sigma_\ell - t'| > u\right) > e^{-2rn^{1/3}}\right] \\ &= \mathbb{P}\left[Q_{n-\ell,t-t'}(|\sigma_0| > u) > e^{-su^2/(n-\ell)}\right] \le Cb^{-3}. \end{aligned}$$

Collecting the estimates

$$\mathbb{P}\left[Q_{(1,n),(0,t)}\left(|\sigma_\ell - t'| > u\right) > h\right] \le Cb^{-3}$$

and from this

$$\begin{aligned} &P_{(1,n),(0,t)}\left(|\sigma_{\lfloor n\gamma\rfloor} - \gamma t| > bn^{2/3}\right) \\ &\qquad \le b^{-3} + \int_{b^{-3}}^1 \mathbb{P}\left[Q_{(1,n),(0,t)}\left(|\sigma_\ell - t'| > u\right) > h\right] dh \le Cb^{-3}. \end{aligned}$$

This completes the proof of the theorem. $\square$

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References

G. Amir, I. Corwin and J. Quastel. Probability distribution of the free energy of the continuum directed random polymer in 1+1 dimensions. *ArXiv Mathematics e-prints* (2000). http://arxiv.org/abs/1003.0443.

J. Baik, P. Deift and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12** (4), 1119–1178 (1999). MR1682248.

M. Balázs, E. Cator and T. Seppäläinen. Cube root fluctuations for the corner growth model associated to the exclusion process. *Electron. J. Probab.* **11**, no. 42, 1094–1132 (electronic) (2006). MR2268539.

M. Balázs, J. Quastel and T. Seppäläinen. Scaling exponent for the Hopf-Cole solution of KPZ/stochastic Burgers. *ArXiv Mathematics e-prints* (2009). http://arxiv.org/abs/0909.4816.

Yu. Baryshnikov. GUEs and queues. *Probab. Theory Related Fields* **119** (2), 256–274 (2001). MR1818248.

E. Cator and P. Groeneboom. Second class particles and cube root asymptotics for Hammersley's process. *Ann. Probab.* **34** (4), 1273–1295 (2006). MR2257647.

F. Comets and N. Yoshida. Brownian directed polymers in random environment. *Comm. Math. Phys.* **254** (2), 257–287 (2005). MR2257647.

D. Dufresne. An affine property of the reciprocal Asian option process. *Osaka J. Math.* **38** (2), 379–381 (2001). MR1833627.

P. L. Ferrari and H. Spohn. Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process. *Comm. Math. Phys.* **265** (1), 1–46 (2006). MR2217296.

J. Gravner, C. A. Tracy and H. Widom. Limit theorems for height fluctuations in a class of discrete space and time growth models. *J. Statist. Phys.* **102** (5-6), 1085–1132 (2001). MR1830441.

K. Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.* **209** (2), 437–476 (2000a). MR1737991.

K. Johansson. Transversal fluctuations for increasing subsequences on the plane. *Probab. Theory Related Fields* **116** (4), 445–456 (2000b). MR1757595.

C. Licea, C. M. Newman and M. S. T. Piza. Superdiffusivity in first-passage percolation. *Probab. Theory Related Fields* **106** (4), 559–591 (1996). MR1421992.

O. Mejane. Upper bound of a volume exponent for directed polymers in a random environment. *Ann. Inst. H. Poincaré Probab. Statist.* **40** (3), 299–308 (2004). MR2060455.

J. Moriarty and N. O'Connell. On the free energy of a directed polymer in a Brownian environment. *Markov Process. Related Fields* **13** (2) (2007). MR2343849.

C. M. Newman and M. S. T. Piza. Divergence of shape fluctuations in two dimensions. *Ann. Probab.* **23** (3), 977–1005 (1995). MR1349159.

N. O'Connell. Directed polymers and the quantum Toda lattice. *ArXiv Mathematics e-prints* (2009). http://arxiv.org/abs/0910.0069.

N. O'Connell and M. Yor. Brownian analogues of Burke's theorem. *Stochastic Process. Appl.* **96** (2), 285–304 (2001). MR1865759.

M. Petermann. *Superdiffusivity of directed polymers in random environment.* PhD dissertation, University of Zürich (2000).

T. Sasamoto and H. Spohn. One-dimensional kardar-parisi-zhang equation: an exact solution and its universality (2010). ArXiv 1002.1883.

T. Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions (2010). ArXiv 0911.2446.

M. V. Wüthrich. Fluctuation results for Brownian motion in a Poissonian potential. *Ann. Inst. H. Poincaré Probab. Statist.* **34** (3), 279–308 (1998a). MR1625875.

M. V. Wüthrich. Superdiffusive behavior of two-dimensional Brownian motion in a Poissonian potential. *Ann. Probab.* **26** (3), 1000–1015 (1998b). MR1634412.