ALEA, Lat. Am. J. Probab. Math. Stat. 8, 59–75 (2011)

Continuous time 'true' self-avoiding random walk on $\mathbb{Z}$

Bálint Tóth and Bálint Vető

Institute of Mathematics, Budapest University of Technology (BME)
Egry József u. 1, H-1111 Budapest, Hungary
E-mail address: `balint@math.bme.hu`
E-mail address: `vetob@math.bme.h`

Abstract. We consider the continuous time version of the 'true' or 'myopic' self-avoiding random walk with site repulsion in $1d$. The Ray−Knight-type method which was applied in (Tóth, 1995) to the discrete time and edge repulsion case is applicable to this model with some modifications. We present a limit theorem for the local time of the walk and a local limit theorem for the displacement.

1. Introduction

1.1. *Historical background.* Let $X(t)$, $t \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ be a nearest neighbour walk on the integer lattice $\mathbb{Z}$ starting from $X(0) = 0$ and denote by $\ell(t, x)$, $(t, x) \in \mathbb{Z}_+ \times \mathbb{Z}$, its local time (that is: its occupation time measure) on sites:

$$\ell(t, x) := \#\{0 \le s \le t : X(s) = x\}$$

where $\#\{\dots\}$ denotes cardinality of the set. The true self-avoiding random walk with site repulsion (STSAW) was introduced in (Amit et al., 1983) as an example for a non-trivial random walk with long memory which behaves qualitatively differently from the usual diffusive behaviour of random walks. It is governed by the evolution rules

$$\begin{aligned}\mathbf{P}\left(X(t+1) = x \pm 1 \,\middle|\, \mathcal{F}_t,\ X(t) = x\right) &= \frac{e^{-\beta\ell(t,x\pm1)}}{e^{-\beta\ell(t,x+1)} + e^{-\beta\ell(t,x-1)}} \\ &= \frac{e^{-\beta(\ell(t,x\pm1)-\ell(t,x))}}{e^{-\beta(\ell(t,x+1)-\ell(t,x))} + e^{-\beta(\ell(t,x-1)-\ell(t,x))}},\end{aligned} \tag{1.1}$$

$$\ell(t+1, x) = \ell(t, x) + 1\!\!1_{\{X(t+1)=x\}}.$$

Received by the editors September 21, 2009; accepted September 20, 2010.
2000 *Mathematics Subject Classification.* 60G50.
Key words and phrases. self-repelling random walk, one dimension, local time, limit theorems, Ray-Knight approach, super-diffusive behaviour.

The extension of this definition to arbitrary dimensions is straightforward. In (Amit et al., 1983), actually, the multidimensional version of the walk was defined. Non-rigorous – nevertheless rather convincing – scaling and renormalization group arguments suggested that:

(1) In three and more dimensions, the walk behaves diffusively with a Gaussian scaling limit of $t^{-1/2}X(t)$ as $t\to\infty$. See e.g. (Amit et al., 1983), (Obukhov and Peliti, 1983) and (Horváth et al., 2010).
(2) In one dimension (that is: the case formally defined above), the walk is superdiffusive with a non-degenerate scaling limit of $t^{-2/3}X(t)$ as $t\to\infty$, but with no hint about the limiting distribution. See (Peliti and Pietronero, 1987), (Tóth, 1999) and (Tóth and Vető, 2008).
(3) The critical dimension is $d=2$ where the Gaussian scaling limit is obtained with logarithmic multiplicative corrections added to the diffusive scaling. See (Amit et al., 1983) and (Obukhov and Peliti, 1983).

These questions are still open. However, the scaling limit in one dimension of a closely related object was clarified in (Tóth, 1995). The true self-avoiding walk with self-repulsion defined in terms of the local times on edges rather than sites is defined as follows:

Let $\widetilde{X}(t)$, $t\in\mathbb{Z}_+:=\{0,1,2,\dots\}$ be yet again a nearest neighbour walk on the integer lattice $\mathbb{Z}$ starting from $\widetilde{X}(0)=0$ and denote now by $\widetilde{\ell}_\pm(t,x)$, $(t,x)\in\mathbb{Z}_+\times\mathbb{Z}$, its local time (that is: occupation time measure) on unoriented edges:

$$\widetilde{\ell}_+(t,x):=\#\{0\le s<t:\{\widetilde{X}(s),\widetilde{X}(s+1)\}=\{x,x+1\}\},$$
$$\widetilde{\ell}_-(t,x):=\#\{0\le s<t:\{\widetilde{X}(s),\widetilde{X}(s+1)\}=\{x,x-1\}\}.$$

Note that $\widetilde{\ell}_+(t,x)=\widetilde{\ell}_-(t,x+1)$. The true self-avoiding random walk with edge repulsion (ETSAW) is governed by the evolution rules

$$\begin{aligned}\mathbf{P}\left(\widetilde{X}(t+1)=x\pm1 \,\middle|\, \mathcal{F}_t,\ \widetilde{X}(t)=x\right)&=\frac{e^{-2\beta\widetilde{\ell}_\pm(t,x)}}{e^{-2\beta\widetilde{\ell}_+(t,x)}+e^{-2\beta\widetilde{\ell}_-(t,x)}}\\&=\frac{e^{-\beta(\widetilde{\ell}_\pm(t,x)-\widetilde{\ell}_\mp(t,x))}}{e^{-\beta(\widetilde{\ell}_+(t,x)-\widetilde{\ell}_-(t,x))}+e^{-\beta(\widetilde{\ell}_-(t,x)-\widetilde{\ell}_+(t,x))}}\\ \widetilde{\ell}_\pm(t+1,x)&=\widetilde{\ell}_\pm(t,x)+1\!\!1_{\{\{\widetilde{X}(t),\widetilde{X}(t+1)\}=\{x,x\pm1\}\}}.\end{aligned}$$

In (Tóth, 1995), a limit theorem was proved for $t^{-2/3}\widetilde{X}(t)$, as $t\to\infty$. Later, in (Tóth and Werner, 1998), a space-time continuous process $\mathbb{R}_+\ni t\mapsto\mathcal{X}(t)\in\mathbb{R}$ was constructed – called the true self-repelling motion (TSRM) – which possessed all the analytic and stochastic properties of an assumed scaling limit of $\mathbb{R}_+\ni t\mapsto\mathcal{X}^{(A)}(t):=A^{-2/3}\widetilde{X}([At])\in\mathbb{R}$. The invariance principle for this model has been clarified in (Newman and Ravishankar, 2006).

A key point in the proof of (Tóth, 1995) is a kind of Ray – Knight-type argument which works for the ETSAW but not for the STSAW. (For the original idea of Ray – Knight theory, see (Knight, 1963) and (Ray, 1963).) Let

$$\widetilde{T}_{\pm,x,h}:=\min\{t\ge0:\widetilde{\ell}_\pm(t,x)\ge h\},\qquad x\in\mathbb{Z},\quad h\in\mathbb{Z}_+$$

be the so called inverse local times and

$$\widetilde{\Lambda}_{\pm,x,h}(y):=\widetilde{\ell}_\pm(\widetilde{T}_{\pm,x,h},y),\qquad x,y\in\mathbb{Z},\quad h\in\mathbb{Z}_+$$

the local time sequence of the walk stopped at the inverse local times. It turns out that, in the ETSAW case, for any fixed $(x,h) \in \mathbb{Z} \times \mathbb{Z}_+$, the process $\mathbb{Z} \ni y \mapsto \widetilde{\Lambda}_{\pm,x,h}(y) \in \mathbb{Z}_+$ is Markovian and it can be thoroughly analyzed.

It is a fact that the similar reduction does not hold for the STSAW. Here, the natural objects are actually slightly simpler to define:

$$T_{x,h} := \min\{t \geq 0 : \ell(t,x) \geq h\}, \qquad x \in \mathbb{Z}, \qquad h \in \mathbb{Z}_+,$$
$$\Lambda_{x,h}(y) := \ell(T_{x,h}, y), \qquad x, y \in \mathbb{Z}, \qquad h \in \mathbb{Z}_+.$$

The process $\mathbb{Z} \ni y \mapsto \Lambda_{x,h}(y) \in \mathbb{Z}_+$ (with fixed $(x,h) \in \mathbb{Z} \times \mathbb{Z}_+$) is not Markovian and thus the Ray−Knight-type of approach fails. Nevertheless, this method works also for the model treated in the present paper.

The main ideas of this paper are similar to those of (Tóth, 1995), but there are essential differences, too. Those parts of the proofs which are the same as in (Tóth, 1995) will not be spelled out explicitly. E.g. the full proof of Theorem 2.6 is omitted altogether. We put the emphasis on those arguments which differ genuinely from (Tóth, 1995). In particular, we present some new coupling arguments.

This paper is organised as follows. First, we describe the model which we will study and present our theorems. In Section 2, we give the proof of Theorem 1.1 in three steps: we introduce the main technical tools, i.e. some auxiliary Markov processes. Then we state technical lemmas which are all devoted to check the conditions of Theorem 2.6 cited from (Tóth, 1995). Finally, we complete the proof using the lemmas. The proof of these lemmas are postponed until Section 4. The proof of Theorem 1.3 is in Section 3.

1.2. *The random walk considered and the main results.* Now, we define a version of true self-avoiding random walk in continuous time, for which the Ray−Knight-type method sketched in the previous section is applicable. Let $X(t)$, $t \in \mathbb{R}_+$ be a *continuous time* random walk on $\mathbb{Z}$ starting from $X(0) = 0$ and having right continuous paths. Denote by $\ell(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{Z}$ its local time (occupation time measure) on sites:

$$\ell(t,x) := |\{s \in [0,t) \,:\, X(s) = x\}|$$

where $|\{\dots\}|$ now denotes Lebesgue measure of the set indicated. Let $w : \mathbb{R} \to (0,\infty)$ be an almost arbitrary rate function. We assume that it is non-decreasing and not constant.

The law of the random walk is governed by the following jump rates and differential equations (for the local time increase):

$$\mathbf{P}\left(X(t+\mathrm{d}t) = x \pm 1 \,\middle|\, \mathcal{F}_t,\ X(t) = x\right) = w(\ell(t,x) - \ell(t,x\pm 1))\,\mathrm{d}t + o(\mathrm{d}t), \tag{1.2}$$
$$\dot{\ell}(t,x) = 1\!\!1_{\{X(t)=x\}} \tag{1.3}$$

with initial conditions

$$X(0) = 0, \qquad \ell(0,x) = 0.$$

The dot in (1.3) denotes time derivative. Note that for the the choice of exponential weight function $w(u) = \exp\{\beta u\}$. This means exactly that conditionally on a jump occurring at the instant t, the random walker jumps to right or left from its actual position with probabilities $e^{-\beta\ell(t,x\pm1)}/(e^{-\beta\ell(t,x+1)} + e^{-\beta\ell(t,x-1)})$, just like in (1.1). It will turn out that in the long run the holding times remain of order one.

Fix $j \in \mathbb{Z}$ and $r \in \mathbb{R}_+$. We consider the random walk $X(t)$ running from $t=0$ up to the stopping time

$$T_{j,r} = \inf\{t \geq 0 : \ell(t,j) \geq r\}, \tag{1.4}$$

which is the inverse local time for our model. Define

$$\Lambda_{j,r}(k) := \ell(T_{j,r}, k) \qquad k \in \mathbb{Z} \tag{1.5}$$

the local time process of X stopped at the inverse local time.

Let

$$\lambda_{j,r} := \inf\{k \in \mathbb{Z} : \Lambda_{j,r}(k) > 0\},$$
$$\rho_{j,r} := \sup\{k \in \mathbb{Z} : \Lambda_{j,r}(k) > 0\}.$$

Fix $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$. Consider the two-sided reflected Brownian motion $W_{x,h}(y)$, $y \in \mathbb{R}$ with starting point $W_{x,h}(x) = h$. Define the times of the first hitting of 0 outside the interval $[0,x]$ or $[x,0]$ with

$$\mathfrak{l}_{x,h} := \sup\{y < 0 \wedge x : W_{x,h}(y) = 0\},$$
$$\mathfrak{r}_{x,h} := \inf\{y > 0 \vee x : W_{x,h}(y) = 0\}$$

where $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$, and let

$$\mathcal{T}_{x,h} := \int_{\mathfrak{l}_{x,h}}^{\mathfrak{r}_{x,h}} W_{x,h}(y)\,\mathrm{d}y. \tag{1.6}$$

The main result of this paper is

Theorem 1.1. *Let $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$ be fixed. Then*

$$A^{-1}\lambda_{\lfloor Ax \rfloor, \lfloor \sqrt{A}\sigma h \rfloor} \Longrightarrow \mathfrak{l}_{0\wedge x,h}, \tag{1.7}$$
$$A^{-1}\rho_{\lfloor Ax \rfloor, \lfloor \sqrt{A}\sigma h \rfloor} \Longrightarrow \mathfrak{r}_{0\vee x,h}, \tag{1.8}$$

and

$$\left(\frac{\Lambda_{\lfloor Ax \rfloor, \lfloor \sqrt{A}\sigma h \rfloor}(\lfloor Ay \rfloor)}{\sigma\sqrt{A}}, \frac{\lambda_{\lfloor Ax \rfloor, \lfloor \sqrt{A}\sigma h \rfloor}}{A} \leq y \leq \frac{\rho_{\lfloor Ax \rfloor, \lfloor \sqrt{A}\sigma h \rfloor}}{A}\right) \Longrightarrow \left(W_{x,h}(y), \mathfrak{l}_{0\wedge x,h} \leq y \leq \mathfrak{r}_{0\vee x,h}\right) \tag{1.9}$$

as $A \to \infty$ where $\sigma^2 = \int_{-\infty}^{\infty} u^2 \rho(\mathrm{d}u) \in (0,\infty)$ with ρ defined by (2.12) and (2.8) later.

Corollary 1.2. *For any $x \in \mathbb{R}$ and $h \geq 0$,*

$$\frac{T_{\lfloor Ax \rfloor, \lfloor \sqrt{A}\sigma h \rfloor}}{\sigma A^{3/2}} \Longrightarrow \mathcal{T}_{x,h}. \tag{1.10}$$

For stating Theorem 1.3, we need some more definitions. It follows from (1.6) that $\mathcal{T}_{x,h}$ has an absolutely continuous distribution. Let

$$\omega(t,x,h) := \frac{\partial}{\partial t}\mathbf{P}\left(\mathcal{T}_{x,h} < t\right) \tag{1.11}$$

be the density of the distribution of $\mathcal{T}_{x,h}$. Define

$$\varphi(t,x) := \int_0^\infty \omega(t,x,h)\,\mathrm{d}h.$$

Theorem 2 of (Tóth, 1995) gives that, for fixed $t > 0$, $\varphi(t, \cdot)$ is a density function, i.e.

$$\int_{-\infty}^{\infty} \varphi(t, x)\, \mathrm{d}x = 1. \tag{1.12}$$

One could expect that $\varphi(t, \cdot)$ is the density of the limit distribution of $X(At)/A^{2/3}$ as $A \to \infty$, but we prove a similar statement for their Laplace transform. We denote by $\hat{\varphi}$ the Laplace transforms of φ:

$$\hat{\varphi}(s, x) := s \int_0^{\infty} e^{-st} \varphi(t, x)\, \mathrm{d}t. \tag{1.13}$$

Theorem 1.3. *Let $s \in \mathbb{R}_+$ be fixed and $\theta_{s/A}$ a random variable of exponential distribution with mean A/s which is independent of the random walk $X(t)$. Then, for almost all $x \in \mathbb{R}$,*

$$A^{2/3} \mathbf{P} \left(X(\theta_{s/A}) = \lfloor A^{2/3} x \rfloor \right) \to \hat{\varphi}(s, x) \tag{1.14}$$

as $A \to \infty$.

From this local limit theorem, the integral limit theorem follows immediately:

$$\lim_{A \to \infty} \mathbf{P} \left(A^{-2/3} X(\theta_{s/A}) < x \right) = \int_{-\infty}^{x} \hat{\varphi}(s, y)\, \mathrm{d}y.$$

2. Ray – Knight construction

The aim of this section is to give a random walk representation of the local time sequence $\Lambda_{j,r}$. Therefore, we introduce auxiliary Markov processes corresponding to each edge of $\mathbb{Z}$. The process corresponding to the edge e is defined in such a way that its value is the difference of local times of $X(T_{j,r})$ on the two vertices adjacent to e where $X(T_{j,r})$ is the process $X(t)$ stopped at an inverse local time. It turns out that the auxiliary Markov processes are independent. Hence, by induction, the sequence of local times can be given as partial sums of independent auxiliary Markov processes. The proof of Theorem 1.1 relies exactly on this observation.

2.1. *The basic construction.* Let

$$\tau(t, k) := \ell(t, k) + \ell(t, k + 1) \tag{2.1}$$

be the local time spent on (the endpoints of) the edge $\langle k, k + 1 \rangle$, $k \in \mathbb{Z}$, and

$$\theta(s, k) := \inf\{t \geq 0 \,:\, \tau(t, k) > s\} \tag{2.2}$$

its inverse. Further on, define

$$\xi_k(s) := \ell(\theta(s, k), k + 1) - \ell(\theta(s, k), k), \tag{2.3}$$

$$\alpha_k(s) := 1\!\!1_{\{X(\theta(s,k))=k+1\}} - 1\!\!1_{\{X(\theta(s,k))=k\}}. \tag{2.4}$$

A crucial observation is that, for each $k \in \mathbb{Z}$, $s \mapsto (\alpha_k(s), \xi_k(s))$ is a Markov process on the state space $\{-1, +1\} \times \mathbb{R}$. The transition rules are

$$\mathbf{P} \left(\alpha_k(t + \mathrm{d}t) = -\alpha_k(t) \,\middle|\, \mathcal{F}_t \right) = w(\alpha_k(t)\xi_k(t))\, \mathrm{d}t + o(\mathrm{d}t), \tag{2.5}$$

$$\dot{\xi}_k(t) = \alpha_k(t), \tag{2.6}$$

with some initial state $(\alpha_k(0), \xi_k(0))$. Furthermore, these processes are independent. In plain words:

(1) $\xi_k(t)$ is the difference of time spent by α_k in the states $+1$ and -1, alternatively, the difference of time spent by the walker on the sites $k+1$ and k;
(2) $\alpha_k(t)$ changes sign with rate $w(\alpha_k(t)\xi_k(t))$ since the walker jumps between k and $k+1$ with these rates.

The common infinitesimal generator of these processes is

$$(Gf)(\pm 1, u) = \pm f'(\pm 1, u) + w(\pm u)\big(f(\mp 1, u) - f(\pm 1, u)\big)$$

where $f'(\pm 1, u)$ is the derivative with respect to the second variable. It is an easy computation to check that these Markov processes are ergodic and their common unique stationary measure is

$$\mu(\pm 1, \mathrm{d}u) = \frac{1}{2Z} e^{-W(u)}\,\mathrm{d}u \tag{2.7}$$

where

$$W(u) := \int_0^u (w(v) - w(-v))\,\mathrm{d}v \quad \text{and} \quad Z := \int_{-\infty}^{\infty} e^{-W(v)}\,\mathrm{d}v. \tag{2.8}$$

Mind that, due to the condition imposed on w (non-decreasing and non-constant),

$$\lim_{|u|\to\infty} \frac{W(u)}{|u|} = \lim_{v\to\infty} (w(v) - w(-v)) > 0, \tag{2.9}$$

and thus $Z < \infty$ and $\mu(\pm 1, \mathrm{d}u)$ is indeed a probability measure on $\{-1, +1\} \times \mathbb{R}$.

Let

$$\beta_\pm(t, k) := \inf\left\{ s \geq 0 : \int_0^s \mathbb{1}_{\{\alpha_k(u) = \pm 1\}}\,\mathrm{d}u \geq t \right\} \tag{2.10}$$

be the inverse local times of $(\alpha_k(t), \xi_k(t))$. With the use of them, we can define the processes

$$\eta_{k,-}(t) := \xi_k(\beta_-(t,k)), \qquad \eta_{k,+}(t) := -\xi_k(\beta_+(t,k)). \tag{2.11}$$

which are also Markovian. By symmetry, the processes with different sign have the same law. The infinitesimal generator of $\eta_{k,\pm}$ is

$$(Hf)(u) = -f'(u) + w(u) \int_u^\infty e^{-\int_u^v w(s)\,\mathrm{d}s} w(v)(f(v) - f(u))\,\mathrm{d}v.$$

It is easy to see that the Markov processes $\eta_{k,\pm}$ are ergodic and their common unique stationary distribution is

$$\rho(\mathrm{d}u) := \frac{1}{Z} e^{-W(u)}\,\mathrm{d}u \tag{2.12}$$

with the notations (2.8). The stationarity of μ is not surprising after (2.7), but a straightforward calculation yields it also.

The main point is the following

Proposition 2.1. *(1) The processes $s \mapsto (\alpha_k(s), \xi_k(s))$, $k \in \mathbb{Z}$ are independent Markov process with the same law given in (2.5)–(2.6). They start from the initial states $\xi_k(0) = 0$ and*

$$\alpha_k(0) = \begin{cases} +1 & \text{if} \quad k < 0, \\ -1 & \text{if} \quad k \geq 0. \end{cases}$$

(2) *The processes $s \mapsto \eta_{k,\pm}(s)$, $k \in \mathbb{Z}$ are independent Markov processes if we consider exactly one of $\eta_{k,+}$ and $\eta_{k,-}$ for each k. The initial distributions are*

$$\mathbf{P}\left(\eta_{k,+}(0) \in A\right) = \begin{cases} Q(0,A) & \textit{if} \quad k \geq 0, \\ \mathbb{1}_{\{0\in A\}} & \textit{if} \quad k < 0, \end{cases} \tag{2.13}$$

$$\mathbf{P}\left(\eta_{k,-}(0) \in A\right) = \begin{cases} \mathbb{1}_{\{0\in A\}} & \textit{if} \quad k \geq 0, \\ Q(0,A) & \textit{if} \quad k < 0. \end{cases} \tag{2.14}$$

2.2. *Technical lemmas.* The lemmas of this subsection descibe the behaviour of the auxiliary Markov processes $\eta_{k,\pm}$. Since they all have the same law, we denote them by η to keep the notation simple, and it means that the statement is true for all $\eta_{k,\pm}$.

Fix $b \in \mathbb{R}$. Define the stopping times

$$\theta_+ := \inf\{t > 0 : \eta(t) \geq b\}, \tag{2.15}$$

$$\theta_- := \inf\{t > 0 : \eta(t) \leq b\}. \tag{2.16}$$

In our lemmas, γ will always be a positive constant, which is considered as being a small exponent, and C will be a finite constant considered as being large. To simplify the notation, we will use the same letter for constants at different points of our proof. The notation does not emphasizes, but their value depend on b.

First, we estimate the exponential moments of θ_- and θ_+.

Lemma 2.2. *There are $\gamma > 0$ and $C < \infty$ such that, for all $y \geq b$,*

$$\mathbf{E}\left(\exp(\gamma\theta_-) \,\middle|\, \eta(0) = y\right) \leq \exp(C(y-b)). \tag{2.17}$$

Lemma 2.3. *There exists $\gamma > 0$ such that*

$$\mathbf{E}\left(\exp(\gamma\theta_+) \,\middle|\, \eta(0) = b\right) < \infty. \tag{2.18}$$

Denote by $P^t = e^{tH}$ the transition kernel of η. For any $x \in \mathbb{R}$, define the probability measure

$$Q(x, \mathrm{d}y) := \begin{cases} \exp(-\int_x^y w(u)\,\mathrm{d}u) w(y)\,\mathrm{d}y & \text{if} \quad y \geq x, \\ 0 & \text{if} \quad y < x, \end{cases}$$

which is the conditional distribution of the endpoint of a jump of η provided that η jumps from x. We show that the Markov process η converges exponentially fast to its stationary distribution ρ defined by (2.12) if the initial distribution is 0 with probability 1 or $Q(0,\cdot)$.

Lemma 2.4. *There are $C < \infty$ and $\gamma > 0$ such that*

$$\left\|P^t(0,\cdot) - \rho\right\| < C\exp(-\gamma t) \tag{2.19}$$

and

$$\left\|Q(0,\cdot)P^t - \rho\right\| < C\exp(-\gamma t). \tag{2.20}$$

We give a bound on the decay of the tails of $P^t(0,\cdot)$ and $Q(0,\cdot)P^t$ uniformly in t.

Lemma 2.5. *There are constants $C < \infty$ and $\gamma > 0$ such that*

$$P^t(0, (x, \infty)) \le Ce^{-\gamma x} \tag{2.21}$$

and

$$Q(0, \cdot)P^t(0, (x, \infty)) \le Ce^{-\gamma x} \tag{2.22}$$

for all $x \ge 0$ and for any $t > 0$ uniformly, i.e. the value of C and γ does not depend on x and t.

We introduce some notation from (Tóth, 1995) and cite a theorem, which will be the main ingredient of our proof. Let $A > 0$ be the scaling parameter, and let

$$S_A(l) = S_A(0) + \sum_{j=1}^{l} \xi_A(j) \qquad l \in \mathbb{N}$$

be a discrete time random walk on $\mathbb{R}_+$ with the law

$$\mathbf{P}\left(\xi_A(l) \in \mathrm{d}x \;\middle|\; S_A(l-1) = y\right) = \pi_A(\mathrm{d}x, y, l)$$

for each $l \in \mathbb{N}$ with

$$\int_{-y}^{\infty} \pi_A(\mathrm{d}x, y, l) = 1.$$

Define the following stopping time of the random walk $S_A(\cdot)$:

$$\omega_{[Ar]} = \inf\{l \ge [Ar] : S_A(l) = 0\}.$$

We give the following theorem without proof, because this is the continuous analog of Theorem 4 in (Tóth, 1995) and its proof is essentially identical to that of the corresponding statement in (Tóth, 1995).

Theorem 2.6. *Suppose that the following conditions hold:*

(1) The step distributions $\pi_A(\cdot, y, l)$ converge exponentially fast as $y \to \infty$ to a common asymptotic distribution π. That is, for each $l \in \mathbb{Z}$,

$$\int_{\mathbb{R}} |\pi_A(\mathrm{d}x, y, l) - \pi(\mathrm{d}x)| < Ce^{-\gamma y}.$$

(2) The asymptotic distribution is symmetric: $\pi(-\mathrm{d}x) = \pi(\mathrm{d}x)$, and its moments are finite, in particular, denote

$$\sigma^2 := \int_{\mathbb{R}} x^2 \pi(\mathrm{d}x). \tag{2.23}$$

(3) Uniform decay of the step distributions: for each $l \in \mathbb{Z}$,

$$\pi_A((x, \infty), y, l) \le Ce^{-\gamma x}.$$

(4) Uniform non-trapping condition: The random walk is not trapped in a bounded domain or in a domain away from the origin. That is, there is $\delta > 0$ such that

$$\int_{\delta}^{\infty} \pi_A(\mathrm{d}x, y, l) > \delta \quad \text{or} \quad \int_{x=\delta}^{\infty} \int_{z=-\infty}^{\infty} \pi_A(\mathrm{d}x - z, y + z, l + 1)\pi_A(\mathrm{d}z, y, l) > \delta \tag{2.24}$$

and

$$\int_{-\infty}^{-(\delta \wedge y)} \pi_A(\mathrm{d}x, y, l) > \delta.$$

Under these conditions, if

$$\frac{S_A(0)}{\sigma\sqrt{A}} \to h,$$

then

$$\left(\frac{\omega_{[Ar]}}{A}, \frac{S_A([Ay])}{\sigma\sqrt{A}} : 0 \le y \le \frac{\omega_{[Ar]}}{A}\right) \Longrightarrow \left(\omega_r^W, |W_y| : 0 \le y \le \omega_r^W \;\middle|\; |W_0| = h\right) \tag{2.25}$$

in $\mathbb{R}_+ \times D[0,\infty)$ *as* $A \to \infty$ *where*

$$\omega_r^W = \inf\{s > 0 : W_s = 0\}$$

with a standard Brownian motion W *and* σ *is given by* (2.23).

2.3. *Proof of Theorem 1.1.* Using the auxiliary Markov processes introduced in Subsection 2.1, we can build up the local time sequence as a random walk. This Ray – Knight-type construction is the main idea of the following proof.

Proof of Theorem 1.1: Fix $j \in \mathbb{Z}$ and $r \in \mathbb{R}_+$. Using the definition (1.5) and the construction of $\eta_{k,\pm}$ (2.1)–(2.11), we can formulate the following recursion for $\Lambda_{j,r}$:

$$\begin{aligned}
&\Lambda_{j,r}(j) = r, && \\
&\Lambda_{j,r}(k+1) = \Lambda_{j,r}(k) + \eta_{k,-}(\Lambda_{j,r}(k)) && \text{if} \quad k \ge j, \\
&\Lambda_{j,r}(k-1) = \Lambda_{j,r}(k) + \eta_{k-1,+}(\Lambda_{j,r}(k)) && \text{if} \quad k \le j.
\end{aligned} \tag{2.26}$$

It means that the processes $(\Lambda_{j,r}(j-k))_{k=0}^{\infty}$ and $(\Lambda_{j,r}(j+k))_{k=0}^{\infty}$ are random walks on $\mathbb{R}_+$, they start from $\Lambda_{j,r}(j) = r$, and the distribution of the following step always depends on the actual position of the walker. In order to apply Theorem 2.6, we rewrite (2.26):

$$\begin{aligned}
\Lambda_{j,r}(j+k) &= h + \sum_{i=0}^{k-1} \eta_{j+i,-}(\Lambda_{j,r}(j+i)) && k = 0,1,2,\ldots, \\
\Lambda_{j,r}(j-k) &= h + \sum_{i=0}^{k-1} \eta_{j-i-1,+}(\Lambda_{j,r}(j-i)) && k = 0,1,2,\ldots.
\end{aligned}$$

The step distributions of this random walks are

$$\pi_A(\mathrm{d}x, y, l) = \begin{cases} P^y(0, \mathrm{d}x) \\ Q(0,\cdot)P^y(\mathrm{d}x) \end{cases}$$

according to (2.13)–(2.14).

The exponential closeness of the step distribution to the stationary distribution is shown by Lemma 2.4. One can see from (2.12) and (2.8) that the distribution ρ is symmetric and it has a non-zero finite variance. Lemma 2.5 gives a uniform exponential bound on the tail of the distributions $P^t(0,\cdot)$ and $Q(0,\cdot)P^t$.

Since we only consider $[\lambda_{j,r}, \rho_{j,r}]$, that is, the time interval until $\Lambda_{j,r}$ hits 0, we can force the walk to jump to 1 in the next step after hitting 0, which does not influence our investigations. It means that $\pi_A(\{1\}, 0, l) = 1$ for $l \in \mathbb{Z}$, and with this, the non-trapping condition (2.24) fulfils. Therefore, Theorem 2.6 is applicable for the forward and the backward walks, and Theorem 1.1 is proved. □

3. The position of the random walker

We turn to the proof of Theorem 1.3. First, we introduce the rescaled distribution

$$\varphi_A(t,x) := A^{2/3}\mathbf{P}\left(X(t) = \lfloor A^{2/3}x \rfloor\right)$$

where $t, x \in \mathbb{R}_+$. We define the Laplace transform of φ_A with

$$\hat{\varphi}_A(s,x) = s\int_0^\infty e^{-st}\varphi_A(t,x)\,\mathrm{d}t, \tag{3.1}$$

which is the position of the random walker at an independent random time of exponential distribution with mean A/s.

We denote by $\hat{\omega}$ the Laplace transforms of ω defined in (1.11) and rewrite (1.13):

$$\begin{aligned}\hat{\omega}(s,x,h) &:= s\int_0^\infty e^{-st}\omega(t,x,h)\,\mathrm{d}t = s\,\mathbf{E}\left(e^{-s\mathcal{T}_{x,h}}\right),\\ \hat{\varphi}(s,x) &= s\int_0^\infty e^{-st}\varphi(t,x)\,\mathrm{d}t = \int_0^\infty \hat{\omega}(s,x,h)\,\mathrm{d}h.\end{aligned}$$

Note that the scaling relations

$$\begin{aligned}\alpha\omega(\alpha t, \alpha^{2/3}x, \alpha^{1/3}h) &= \omega(t,x,h),\\ \alpha^{2/3}\hat{\varphi}(\alpha^{-1}s, \alpha^{2/3}x) &= \hat{\varphi}(s,x)\end{aligned} \tag{3.2}$$

hold because of the scaling property of the Brownian motion.

Proof of Theorem 1.3: The first observation for the proof is the identity

$$\mathbf{P}\left(X(t) = k\right) = \int_{h=0}^\infty \mathbf{P}\left(T_{k,h} \in (t, t+\mathrm{d}h)\right), \tag{3.3}$$

which follows from (1.4). If we insert it to the definition of $\hat{\varphi}_A$ (3.1), then we get

$$\begin{aligned}\hat{\varphi}_A(s,x) &= sA^{-1/3}\int_0^\infty e^{-st/A}\mathbf{P}\left(X(t) = \lfloor A^{2/3}x\rfloor\right)\mathrm{d}t\\ &= sA^{-1/3}\int_0^\infty e^{-st/A}\int_{h=0}^\infty \mathbf{P}\left(T_{\lfloor A^{2/3}x\rfloor,h} \in (t,t+\mathrm{d}h)\right)\mathrm{d}t\\ &= sA^{-1/3}\int_0^\infty \mathbf{E}\left(e^{-sT_{\lfloor A^{2/3}x\rfloor,h}/A}\right)\mathrm{d}h\end{aligned} \tag{3.4}$$

using (3.3). Defining

$$\hat{\omega}_A(s,x,h) = s\mathbf{E}\left(\exp(-sT_{\lfloor A^{2/3}x\rfloor,\lfloor A^{1/3}\sigma h\rfloor}/(\sigma A))\right)$$

gives us

$$\hat{\varphi}_A(s,x) = \int_0^\infty \hat{\omega}_A(\sigma s,x,h)\,\mathrm{d}h \tag{3.5}$$

from (3.4). From Corollary 1.2, it follows that, for any $s > 0$, $x \geq 0$ and $h > 0$,

$$\hat{\omega}_A(s,x,h) \to \hat{\omega}(s,x,h).$$

Applying Fatou's lemma in (3.5), one gets

$$\liminf_{A\to\infty} \hat{\varphi}_A(s,x) \geq \int_0^\infty \hat{\omega}(\sigma s,x,h)\,\mathrm{d}h = \sigma^{2/3}\hat{\varphi}(s,\sigma^{2/3}x), \tag{3.6}$$

where we used (3.2) in the last equation. A consequence of (1.12), (3.6) integrated and a second application of Fatou's lemma yield

$$1 = \int_{-\infty}^{\infty} \hat{\varphi}(s,x)\,\mathrm{d}x \le \int_{-\infty}^{\infty} \liminf_{A\to\infty} \hat{\varphi}_A(s,x)\,\mathrm{d}x \le \liminf_{A\to\infty} \int_{-\infty}^{\infty} \hat{\varphi}_A(s,x)\,\mathrm{d}x = 1,$$

which gives that, for fixed $s \in \mathbb{R}_+$, $\hat{\varphi}_A(s,x) \to \hat{\varphi}(s,x)$ holds for almost all $x \in \mathbb{R}$, indeed. □

4. Proof of lemmas

4.1. *Exponential moments of the return times.*

Proof of Lemma 2.2: Consider the Markov process $\zeta(t)$ which decreases with constant speed 1, it has upwards jumps with homogeneous rate $w(-b)$, and the distribution of the size of a jump is the same as that of η, provided that the jump starts from b. In other words, the infinitesimal generator of ζ is

$$(Zf)(u) = -f'(u) + w(-b) \int_0^{\infty} e^{-\int_0^v w(b+s)\,\mathrm{d}s} w(b+v)(f(u+v) - f(u))\,\mathrm{d}v.$$

Note that, by the monotonicity of w, η and ζ can be coupled in such a way that they start from the same position and, as long as $\eta \ge b$ holds, $\zeta \ge \eta$ is true almost surely. It means that it suffices to prove (2.17) with

$$\theta'_- := \inf\{t > 0 : \zeta(t) \le b\} \tag{4.1}$$

instead of θ_-. But the transitions of ζ are homogeneous in space, which yields that (2.17) follows from the finiteness of

$$\mathbf{E}\left(\exp(\gamma\theta'_-) \,\middle|\, \zeta(0) = b+1\right). \tag{4.2}$$

In addition to this, ζ is a supermartingale with stationary increments, which gives us

$$\mathbf{E}\left(\zeta(t)\right) = b + 1 - ct$$

with some $c > 0$, if the initial condition is $\zeta(0) = b+1$. For $\alpha \in \left(-\infty, \lim_{u\to\infty} \frac{W(u)}{u}\right)$ (c.f. (2.9)), the expectation

$$\log \mathbf{E}\left(e^{\alpha(\zeta(t)-\zeta(0))}\right)$$

is finite, and negative for some $\alpha > 0$. Hence, the martingale

$$M(t) = \exp\left(\alpha(\zeta(t) - \zeta(0)) - t \log \mathbf{E}\left(e^{\alpha(\zeta(1)-\zeta(0))}\right)\right) \tag{4.3}$$

stopped at θ'_- gives that the expectation in (4.2) is finite with

$$\gamma = -\log \mathbf{E}\left(e^{\alpha(\zeta(1)-\zeta(0))}\right)$$

. □

Proof of Lemma 2.3: First, we prove for negative b, more precisely, for which $w(-b) > w(b)$. In this case, define the homogeneous process κ with $\kappa(0) = b$ and generator

$$Kf(u) = -f'(u) + w(-b) \int_0^{\infty} e^{-w(b)s} w(b)(f(u+s) - f(u))\,\mathrm{d}s.$$

It is easy to see that there is a coupling of η and κ, for which $\eta \geq \kappa$ as long as $\eta \leq b$. Therefore, it is enough to show (2.18) with

$$\theta'_+ := \inf\{t > 0 : \kappa(t) \geq b\}$$

instead of θ_+.

But κ is a submartingale with stationary increments, for which

$$\log \mathbf{E}\left(e^{\alpha(\kappa(t)-\kappa(0))}\right)$$

is finite if $\alpha \in (-\infty, w(b))$, and negative for some $\alpha < 0$. The statement follows from the same idea as in the proof of Lemma 2.2.

Now, we prove the lemma for the remaining case. Fix b, for which we already know (2.18), and chose $b_1 > b$ arbitrarily. We start η from $\eta(0) = b_1$, and we decompose its trajectory into independent excursions above and below b, alternatingly. Let

$$Y_0 := \inf\{t \geq 0 : \eta(t) \leq b\}, \tag{4.4}$$

and by induction, define

$$X_k := \inf\left\{t > 0 : \eta\left(\sum_{j=1}^{k-1} X_j + \sum_{j=0}^{k-1} Y_j + t\right) \geq b\right\}, \tag{4.5}$$

$$Y_k := \inf\left\{t \geq 0 : \eta\left(\sum_{j=1}^{k} X_j + \sum_{j=0}^{k-1} Y_j + t\right) \leq b\right\} \tag{4.6}$$

if $k = 1, 2, \dots$. Note that $(X_k, Y_k)_{k=1,2,\dots}$ is an i.i.d. sequence of pairs of random variables. Finally, let

$$Z_k := X_k + Y_k \qquad k = 1, 2, \dots. \tag{4.7}$$

With this definition, the Z_k's are the lengths of the epochs in a renewal process. Lemma 2.2 tells us that Y_0 has finite exponential moment. The same holds for $X_1, X_2, \dots$ because of the first part of this proof for the case of small b. Note that the distribution of the upper endpoint of a jump of η conditionally given that η jumps above b is exactly $Q(b, \cdot)$. Since $Q(b, \cdot)$ decays exponentially fast, we can use Lemma 2.2 again to conclude that $\mathbf{E}(\exp(\gamma Y_k)) < \infty$ for $\gamma > 0$ small enough. Define

$$\nu_t := \max\left\{n \geq 0 : \sum_{k=1}^{n} Z_k \leq t\right\} \tag{4.8}$$

in the usual way. The following decomposition is true:

$$\begin{aligned}\mathbf{P}&\left(\frac{\sum_{k=1}^{\nu_t+1} Y_k}{t} < \varepsilon\right)\\ &\leq \mathbf{P}\left(\frac{\nu_t+1}{t} < \frac{1}{2}\frac{1}{\mathbf{E}(Z_1)}\right) + \mathbf{P}\left(\frac{\sum_{k=1}^{\nu_t+1} Y_k}{t} < \varepsilon, \frac{\nu_t+1}{t} \geq \frac{1}{2}\frac{1}{\mathbf{E}(Z_1)}\right).\end{aligned} \tag{4.9}$$

Lemma 4.1 of (van den Berg and Tóth, 1991) gives a large deviation principle for the renewal process ν_t, hence

$$\mathbf{P}\left(\frac{\nu_t+1}{t} < \frac{1}{2}\frac{1}{\mathbf{E}(Z_1)}\right) \leq \mathbf{P}\left(\frac{\nu_t}{t} < \frac{1}{2}\frac{1}{\mathbf{E}(Z_1)}\right) < e^{-\gamma t} \tag{4.10}$$

with some $\gamma > 0$. For the second term on the right-hand side in (4.9),

$$
\begin{aligned}
\mathbf{P}&\left(\frac{\sum_{k=1}^{\nu_t+1} Y_k}{t} < \varepsilon, \frac{\nu_t+1}{t} \geq \frac{1}{2}\frac{1}{\mathbf{E}(Z_1)}\right) \\
&= \mathbf{P}\left(\frac{\sum_{k=1}^{\nu_t+1} Y_k}{\nu_t+1} < \varepsilon\frac{t}{\nu_t+1}, \frac{\nu_t+1}{t} \geq \frac{1}{2}\frac{1}{\mathbf{E}(Z_1)}\right) \\
&\leq \mathbf{P}\left(\frac{\sum_{k=1}^{\nu_t+1} Y_k}{\nu_t+1} < 2\varepsilon\mathbf{E}(Z_1), \frac{\nu_t+1}{t} \geq \frac{1}{2}\frac{1}{\mathbf{E}(Z_1)}\right) \\
&\leq \max_{n \geq \frac{1}{2}\frac{1}{\mathbf{E}(Z_1)}t} \mathbf{P}\left(\frac{\sum_{k=1}^{n} Y_k}{n} < 2\varepsilon\mathbf{E}(Z_1)\right),
\end{aligned}
\tag{4.11}
$$

which is exponentially small for some $\varepsilon > 0$ by standard large deviation theory, and the same holds for the probability estimated is (4.9), which means that η spends at least εt time above b with overwhelming probability.

The inequality

$$
\mathbf{P}\left(\theta_+ > t \,\middle|\, \eta(0) = b_1\right) \leq \mathbf{P}\left(\sum_{k=1}^{\nu_t+1} Y_k < \varepsilon t\right) + \mathbf{P}\left(\theta_+ > t \,\middle|\, \eta(0) = b_1, \sum_{k=1}^{\nu_t+1} Y_k > \varepsilon t\right)
$$

is obvious. The first term on the right-hand side is exponentially small by (4.9)–(4.11). In order to bound the second term, denote by $J(t)$ the number of jumps when $\eta(s) \geq b$. The condition $\sum_{k=1}^{\nu_t+1} Y_k > \varepsilon t$ means that this is the case in an at least ε portion of $[0,t]$. The rate of these jumps are at least $w(-b)$ by the monotonicity of w. Note that $J(t)$ dominates stochastically a Poisson random variable $L(t)$ with mean $w(-b)t$. Hence,

$$
\mathbf{P}\left(J(t) < \frac{1}{2}w(-b)t\right) \leq \mathbf{P}\left(L(t) < \frac{1}{2}w(-b)t\right) < e^{-\gamma t} \tag{4.12}
$$

for t large enough with some $\gamma > 0$ by a standard large deviation estimate.

Note that Q is also monotone in the sense that

$$
\int_{b_1}^{\infty} Q(x_1, \mathrm{d}y) < \int_{b_1}^{\infty} Q(x_2, \mathrm{d}y)
$$

if $x_1 < x_2$. Therefore, a jump of η, which starts above b, exits $(-\infty, b_1]$ with probability at least

$$
r = \int_{b_1}^{\infty} Q(b, \mathrm{d}y) > 0.
$$

Finally,

$$
\begin{aligned}
\mathbf{P}&\left(\theta_+ > t \,\middle|\, \eta(0) = b_1, \sum_{k=1}^{\nu_t+1} Y_k > \varepsilon t\right) \\
&\leq \mathbf{P}\left(J(t) < \frac{1}{2}w(-b)\varepsilon t\right) \\
&\quad + \mathbf{P}\left(\theta_+ > t \,\middle|\, J(t) \geq \frac{1}{2}w(-b)\varepsilon t, \eta(0) = b_1, \sum_{k=1}^{\nu_t+1} Y_k > \varepsilon t\right) \\
&\leq e^{-\gamma t} + (1-r)^{\frac{1}{2}w(-b)\varepsilon t}
\end{aligned}
$$

by (4.12), which is an exponential decay, as required. □

4.2. *Exponential convergence to the stationarity.*

Proof of Lemma 2.4: First, we prove (2.19). We couple two copies of η, say η_1 and η_2. Suppose that

$$\eta_1(0) = 0 \qquad \text{and} \qquad \mathbf{P}\left(\eta_2(0) \in A\right) = \rho(A).$$

Their distribution after time t are obviously $P^t(0,\cdot)$ and ρ, respectively. We use the standard coupling lemma to estimate their variation distance:

$$\left\|P^t(0,\cdot) - \rho\right\| \le \mathbf{P}\left(T > t\right)$$

where T is the random time when the two processes merge.

Assume that $\eta_1 = x_1$ and $\eta_2 = x_2$ with fixed numbers $x_1, x_2 \in \mathbb{R}$. Then there is a coupling where the rate of merge is

$$c(x_1,x_2) := w(-x_1 \vee x_2) \exp\left(-\int_{x_1 \wedge x_2}^{x_1 \vee x_2} w(z)\,\mathrm{d}z\right).$$

Consider the interval $I_b = (-b, b)$ where b will be chosen later appropriately. If $\eta_1 = x_1$ and $\eta_2 = x_2$ where $x_1, x_2 \in I_b$, then for the rate of merge

$$c(x_1,x_2) \ge w(-b) \exp\left(-\int_{-b}^{b} w(z)\,\mathrm{d}z\right) =: \beta(b) > 0 \tag{4.13}$$

holds if $w(x) > 0$ for all $x \in \mathbb{R}$.

Let ϑ be the time spent in I_b, more precisely,

$$\vartheta_i(t) := \left|\{0 \le s \le t : \eta_i(s) \in I_b\}\right| \qquad i = 1,2,$$
$$\vartheta_{12}(t) := \left|\{0 \le s \le t : \eta_1(s) \in I_b, \eta_2(s) \in I_b\}\right|.$$

The estimate

$$\mathbf{P}\left(T > t\right) \le \mathbf{P}\left(\vartheta_{12}(t) < \frac{t}{2}\right) + \mathbf{P}\left(T > t \,\middle|\, \vartheta_{12}(t) \ge \frac{t}{2}\right)$$

is clearly true. Note that

$$\mathbf{P}\left(T > t \,\middle|\, \vartheta_{12}(t) \ge \frac{t}{2}\right) \le \exp\left(-\frac{1}{2}\beta(b)t\right)$$

follows from (4.13).

By the inclusion relation

$$\left\{\vartheta_{12}(t) < \frac{t}{2}\right\} \subset \left\{\vartheta_1(t) < \frac{3}{4}t\right\} \cup \left\{\vartheta_2 < \frac{3}{4}t\right\}, \tag{4.14}$$

it suffices to prove that the tails of $\mathbf{P}\left(\vartheta_i(t) < \frac{3}{4}t\right)$ decay exponentially $i = 1,2$, if b is large enough.

We will show that

$$\mathbf{P}\left(\frac{\left|\{0 \le s \le t : \eta(s) < b\}\right|}{t} < \frac{7}{8}\right) \le e^{-\gamma t}. \tag{4.15}$$

A similar statement can be proved for the time spent above $-b$, therefore another inclusion relation like (4.14) gives the lemma.

First, we verify that the first hitting of level b

$$\inf\{s > 0 : \eta_i(s) = b\}$$

has finite exponential moment, hence, it is negligible with overwhelming probability and we can suppose that $\eta_i(0) = b$. Indeed, for any fixed $\varepsilon > 0$, the measures ρ and $Q(b, \cdot)$ assign exponentially small weight to the complement of the interval $[-\varepsilon t, \varepsilon t]$ as $t \to \infty$. From now on, we suppress the subscript of η_i, we forget about the initial values, and assume only that $\eta(0) \in [-\varepsilon t, \varepsilon t]$.

If $\eta(0) \in [b, \varepsilon t]$, then recall the proof Lemma 2.2. There, we could majorate η with a homogeneous process ζ. If we define

$$a := \mathbf{E}\left(\theta'_- \,\middle|\, \zeta(0) = b + 1\right)$$

with the notation (4.1), which is finite by Lemma 2.2, then from a large deviation principle,

$$\mathbf{P}\left(\theta_-(t) > 2a\varepsilon t \,\middle|\, \eta(0) \in [b, \varepsilon t]\right) \le \mathbf{P}\left(\theta'_-(t) > 2a\varepsilon t \,\middle|\, \eta(0) \in [b, \varepsilon t]\right) < e^{-\gamma t} \tag{4.16}$$

with some $\gamma > 0$.

If $\eta(0) \in [-\varepsilon t, b]$, then we can neglect that piece of the trajectory of η which falls into the interval $[0, \theta_+]$, because without this, $\vartheta(t)$ decreases and the bound on (4.15) becomes stronger. Since η jumps at θ_+ a.s. and the distribution of $\eta(\theta_+)$ is $Q(b, \cdot)$, we can use the previous observations concerning the case $\eta(0) \in [b, \varepsilon t]$.

Using (4.16), it is enough to prove that

$$\mathbf{P}\left(\frac{|\{0 \le s \le t : \eta(s) < b\}|}{t} < \frac{7}{8} + 2a\varepsilon\right) \le e^{-\gamma t}$$

with the initial condition $\eta(0) = b$ where the value of b is not specified yet. We introduce X_k, Y_k, Z_k and ν_t as in (4.5)–(4.8) with $Y_0 \equiv 0$. The only difference is that here we want to ensure a given portion of time spent below b with high probability with the appropriate choice of b. With the same idea as in the proof of Lemma 2.3 in (4.9)–(4.11), we can show that

$$\mathbf{P}\left(\frac{\sum_{k=1}^{\nu_t+1} X_k}{t} \le \frac{7}{8} + 2a\varepsilon\right)$$

is exponentially small by large deviation theory if we choose b large enough to set $\mathbf{E}(X_1)/\mathbf{E}(Z_1)$ (the expected portion of time spent below b) sufficiently close to 1. With this, the proof of (2.19) is complete, that of (2.20) is similar. □

4.3. *Decay of the transition kernel.*

Proof of Lemma 2.5: We return to the idea that the partial sums of Z_k's form a renewal process. Remember the definitions (4.5)–(4.8). This proof relies on the estimate

$$|\eta(t)| \le Z_{\nu_t+1},$$

which is true, because the process η can decrease with speed at most 1. Therefore, it suffices to prove the exponential decay of the tail of Z_{ν_t+1}.

Define the *renewal measure* with

$$U(A) := \sum_{n=0}^{\infty} \mathbf{P}\left(\sum_{k=1}^{n} Z_k \in A\right)$$

for any $A \subset \mathbb{R}$. We consider the *age* and the *residual waiting time*

$$A_t := t - \sum_{k=1}^{\nu_t} Z_k,$$
$$R_t := \sum_{k=1}^{\nu_t+1} Z_k - t$$

separately. For the distribution of the former $H(t,x) := \mathbf{P}(A_t > x)$, the renewal equation

$$H(t,x) = (1 - F(t))1\!\!1_{\{t>x\}} + \int_0^t H(t-s,x)\,\mathrm{d}F(s) \tag{4.17}$$

holds where $F(x) = \mathbf{P}(Z_1 < x)$. (4.17) can be deduced by conditioning on the time of the first renewal, Z_1. From Theorem (4.8) in (Durrett, 1996), it follows that

$$H(t,x) = \int_0^t (1 - F(t-s))1\!\!1_{\{t-s>x\}} U(\mathrm{d}s). \tag{4.18}$$

As explained after (4.7), Lemma 2.2 and Lemma 2.3 with $b = 0$ together imply that $1 - F(x) \le Ce^{-\gamma x}$ with some $C < \infty$ and $\gamma > 0$. On the other hand,

$$U([k,k+1]) \le U([0,1])$$

is true, because, in the worst case, there is a renewal at time k. Otherwise, the distribution of renewals in $[k,k+1]$ can be obtained by shifting the renewals in $[0,1]$ with R_k. We can see from (4.18) by splitting the integral into segments with unit length that

$$H(t,x) \le U([0,1]) \sum_{k=\lfloor x \rfloor}^{\infty} Ce^{-\gamma k},$$

which is uniform in $t > 0$.

With the equation

$$\{R_t > x\} = \{A_{t+x} \ge x\} = \{\text{no renewal in } (t,t+x]\},$$

a similar uniform exponential bound can be deduced for the tail $\mathbf{P}(R_t > x)$. Since $Z_{\nu_t+1} = A_t + R_t$, the proof is complete. □

Acknowledgements. This research was partially supported by the OTKA (Hungarian National Research Fund) grants K 60708 (for B. T. and B. V.) and TS 49835 (for B. V.). B. V. thanks the kind hospitality of the Erwin Schrödinger Institute (Vienna) where part of this work was done.

References

D. J. Amit, G. Parisi and L. Peliti. Asymptotic behavior of the "true" self-avoiding walk. *Phys. Rev. B (3)* **27** (3), 1635–1645 (1983). MR690540.

M. van den Berg and B. Tóth. Exponential estimates for the Wiener sausage. *Probab. Theory Related Fields* **88** (2), 249–259 (1991). MR1096482.

R. Durrett. *Probability: theory and examples*. Duxbury Press, Belmont, CA, second edition (1996). MR1609153.

I. A. Horváth, B. Tóth and B. Vető. Diffusive limits for "true" (or myopic) self-avoiding random walks and self-repellent Brownian polymers in $d \ge 3$. *ArXiv Mathematics e-prints* (2010). http://arxiv.org/abs/1009.0401.

F. B. Knight. Random walks and a sojourn density process of Brownian motion. *Trans. Amer. Math. Soc.* **109**, 56–86 (1963). MR0154337.

C. M. Newman and K. Ravishankar. Convergence of the Tóth lattice filling curve to the Tóth-Werner plane filling curve. *ALEA Lat. Am. J. Probab. Math. Stat.* **1**, 333–345 (2006). MR2249660.

S. P. Obukhov and L. Peliti. Renormalisation of the "true" self-avoiding walk. *J. Phys. A* **16** (5), L147–L151 (1983). MR712594.

L. Peliti and L. Pietronero. Random walks with memory. *La Rivista del Nuovo Cimento* **10**, 1–33 (1987).

D. Ray. Sojourn times of diffusion processes. *Illinois J. Math.* **7**, 615–630 (1963). MR0156383.

B. Tóth. The "true" self-avoiding walk with bond repulsion on **Z**: limit theorems. *Ann. Probab.* **23** (4), 1523–1556 (1995). MR1379158.

B. Tóth. Self-interacting random motions—a survey. In *Random walks (Budapest, 1998)*, volume 9 of *Bolyai Soc. Math. Stud.*, pages 349–384. János Bolyai Math. Soc., Budapest (1999). MR1752900.

B. Tóth and B. Vető. Self-repelling random walk with directed edges on **Z**. *Electron. J. Probab.* **13**, no. 62, 1909–1926 (2008). MR2453550.

B. Tóth and W. Werner. The true self-repelling motion. *Probab. Theory Related Fields* **111** (3), 375–452 (1998). MR1640799.