ALEA, Lat. Am. J. Probab. Math. Stat. 8, 77–94 (2011)

Central limit theorems for Hilbert-space valued random fields satisfying a strong mixing condition

Cristina Tone

Department of Mathematics, University of Louisville, 328 NS, Louisville, Kentucky 40292
E-mail address: cristina.tone@louisville.edu

Abstract. In this paper we study the asymptotic normality of the normalized partial sum of a Hilbert-space valued strictly stationary random field satisfying the interlaced ρ'-mixing condition.

1. Introduction

In the literature about Hilbert-valued random sequences under mixing conditions, progress has been made by Mal′tsev and Ostrovskiĭ (1982), Merlevède (2003), and Merlevède et al. (1997). Dedecker and Merlevède (2002) established a central limit theorem and its weak invariance principle for Hilbert-valued strictly stationary sequences under a projective criterion. In this way, they recovered the special case of Hilbert-valued martingale difference sequences, and under a strong mixing condition involving the whole past of the process and just one future observation at a time, they gave the nonergodic version of the result of Merlevède et al. (1997). Later on, Merlevède (2003) proved a central limit theorem for a Hilbert-space valued strictly stationary, strongly mixing sequence, where the mixing coefficients involve the whole past of the process and just two future observations at a time, by using the Bernstein blocking technique and approximations by martingale differences.

This paper will present a central limit theorem for strictly stationary Hilbert-space valued random fields satisfying the ρ'-mixing condition. We proceed by proving in Theorem 3.1 a central limit theorem for a ρ'-mixing strictly stationary random field of real-valued random variables, by the use of the Bernstein blocking technique. Next, in Theorem 3.2 we extend the real-valued case to a random field of m-dimensional random vectors, $m \geq 1$, satisfying the same mixing condition. Finally, being able to prove the tightness condition in Theorem 3.3, we extend the finite-dimensional case even further to a (infinite-dimensional) Hilbert space-valued strictly stationary random field in the presence of the ρ'-mixing condition.

Received by the editors December 30, 2009; accepted November 23, 2010.
2000 *Mathematics Subject Classification.* 60G60, 60B12, 60F05.
Key words and phrases. Central limit theorem, ρ'-mixing, Hilbert-space valued random fields, Bernstein's blocking argument, tightness, covariance operator.

2. Preliminary Material

For the clarity of the proofs of the three theorems mentioned above, relevant definitions, notations and basic background information will be given first.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose H is a separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_H$. Let $\mathcal{H}$ be the σ-field generated by the class of all open subsets of H. Let $\{e_k\}_{k \geq 1}$ be an orthonormal basis for the Hilbert space H. Then for every $x \in H$, we denote by x_k the kth coordinate of x, defined by $x_k = \langle x, e_k \rangle$, $k \geq 1$. Also, for every $x \in H$ and every $N \geq 1$ we set

$$r_N^2(x) = \sum_{k=N}^{\infty} x_k^2 = \sum_{k=N}^{\infty} \langle x, e_k \rangle^2.$$

For any given H-valued random variable X with $EX = 0_H$ and $E\|X\|_H^2 < \infty$, represent X by

$$X = \sum_{k=1}^{\infty} X_k e_k,$$

where $X_1, X_2, X_3, \ldots$ are real-valued random variables having $EX_k = 0$ and $EX_k^2 < \infty$, $\forall\, k \geq 1$ (in fact, $\sum_{k=1}^{\infty} EX_k^2 = E\|X\|_H^2 < \infty$). Then the "covariance operator" (defined relative to the given orthonormal basis) for the (centered) H-valued random variable X can be thought of as represented by the $\mathbb{N} \times \mathbb{N}$ "covariance matrix" $\Sigma := (\sigma_{ij}, i \geq 1, j \geq 1)$, where $\sigma_{ij} := EX_iX_j$.

Lemma 2.1. *Let $\mathcal{P}_0$ be a class of probability measures on $(H, \mathcal{H})$ satisfying the following conditions:*

$$\sup_{P \in \mathcal{P}_0} \int_H r_1^2(x) dP(x) < \infty, \ \textit{and}$$

$$\lim_{N \to \infty} \sup_{P \in \mathcal{P}_0} \int_H r_N^2(x) dP(x) = 0.$$

Then $\mathcal{P}_0$ is tight.

For the proof of the lemma, see Laha and Rohatgi (1979), Theorem 7.5.1.

For any two σ-fields $\mathcal{A}$, $\mathcal{B} \subseteq \mathcal{F}$, define now the strong mixing coefficient

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|,$$

and the maximal coefficient of correlation

$$\rho(\mathcal{A}, \mathcal{B}) := \sup |Corr(f, g)|, \ f \in L^2_{\text{real}}(\mathcal{A}), \ g \in L^2_{\text{real}}(\mathcal{B}).$$

Suppose d is a positive integer and $X := (X_k, k \in \mathbb{Z}^d)$ is a strictly stationary random field. In this context, for each positive integer n, define the following quantity:

$$\alpha(n) := \alpha(X, n) := \sup \alpha(\sigma(X_k, k \in Q), \sigma(X_k, k \in S)),$$

where the supremum is taken over all pairs of nonempty, disjoint sets Q, $S \subset \mathbb{Z}^d$ with the following property: There exist $u \in \{1, 2, \ldots, d\}$ and $j \in \mathbb{Z}$ such that $Q \subset \{k := (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d : k_u \leq j\}$ and $S \subset \{k := (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d : k_u \geq j + n\}$.

The random field $X := (X_k, k \in \mathbb{Z}^d)$ is said to be "strongly mixing" (or "α-mixing") if $\alpha(n) \to 0$ as $n \to \infty$.

Also, for each positive integer n, define the following quantity:

$$\rho'(n) := \rho'(X, n) := \sup \rho(\sigma(X_k, k \in Q), \sigma(X_k, k \in S)),$$

where the supremum is taken over all pairs of nonempty, finite disjoint sets Q, $S \subset \mathbb{Z}^d$ with the following property: There exist $u \in \{1, 2, \dots, d\}$ and nonempty disjoint sets A, $B \subset \mathbb{Z}$, with $dist(A, B) := \min_{a \in A, b \in B} |a - b| \geq n$ such that $Q \subset \{k := (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d : k_u \in A\}$ and $S \subset \{k := (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d : k_u \in B\}$.

The random field $X := (X_k, k \in \mathbb{Z}^d)$ is said to be "ρ'-mixing" if $\rho'(n) \to 0$ as $n \to \infty$.

Again, suppose d is a positive integer, and suppose $X := (X_k, k \in \mathbb{Z}^d)$ is a strictly stationary Hilbert-space random field. Elements of $\mathbb{N}^d$ will be denoted by $L := (L_1, L_2, \dots, L_d)$. For any $L \in \mathbb{N}^d$, define the "rectangular sum":

$$S_L = S(X, L) := \sum_k X_k,$$

where the sum is taken over all d-tuples $k := (k_1, k_2, \dots, k_d) \in \mathbb{N}^d$ such that $1 \leq k_u \leq L_u$ for all $u \in \{1, 2, \dots, d\}$. Thus $S(X, L)$ is the sum of $L_1 \cdot L_2 \cdot \dots \cdot L_d$ of the $X_k's$.

Proposition 2.2. *Suppose d is a positive integer.*

(I) Suppose $(a(k),\ k \in \mathbb{N}^d)$ is an array of real (or complex) numbers and b is a real (or complex) number. Suppose that for every $u \in \{1, 2, \dots, d\}$ and every sequence $\left(L^{(n)},\ n \in \mathbb{N}\right)$ of elements of $\mathbb{N}^d$ such that $L_u^{(n)} = n$ for all $n \geq 1$, and $L_v^{(n)} \to \infty$ as $n \to \infty$, $\forall\ v \in \{1, 2, \dots, d\} \setminus \{u\}$, one has that $\lim_{n\to\infty} a\left(L^{(n)}\right) = b$. Then $a(L) \to b$ as $\min\{L_1, L_2, \dots, L_d\} \to \infty$.

(II) Suppose $(\mu(k),\ k \in \mathbb{N}^d)$ is an array of probability measures on $(S, \mathcal{S})$, where (S, d) is a complete separable metric space and $\mathcal{S}$ is the σ-field on S generated by the open balls in S in the given metric d. Suppose ν is a probability measure on $(S, \mathcal{S})$ and that for every $u \in \{1, 2, \dots, d\}$ and every sequence $(L^{(n)},\ n \in \mathbb{N})$ of elements of $\mathbb{N}^d$ such that $L_u^{(n)} = n$ for all $n \geq 1$, and $L_v^{(n)} \to \infty$ as $n \to \infty$, $\forall\ v \in \{1, 2, \dots, d\} \setminus \{u\}$, one has that $\mu\left(L^{(n)}\right) \Rightarrow \nu$. Then $\mu(L) \Rightarrow \nu$ as $\min\{L_1, L_2, \dots, L_d\} \to \infty$.

Let us specify that the proof of this proposition follows exactly the proof given in Bradley (2007), A2906 Proposition (parts (I) and (III)) with just a small, insignificant change.

For each $n \geq 1$ and each $\lambda \in [-\pi, \pi]$, define now the Fejér kernel, $K_{n-1}(\lambda)$ by:

$$K_{n-1}(\lambda) := \frac{1}{n} \left| \sum_{j=0}^{n-1} e^{ij\lambda} \right|^2 = \frac{\sin^2(n\lambda/2)}{n \sin^2(\lambda/2)}. \tag{2.1}$$

Elements of $[-\pi, \pi]^d$ will be denoted by $\vec{\lambda} := (\lambda_1, \lambda_2, \dots, \lambda_d)$. For each $L \in \mathbb{N}^d$ define the "multivariate Fejér kernel" $G_L : [-\pi, \pi]^d \to [0, \infty)$ by:

$$G_L(\vec{\lambda}) := \prod_{u=1}^{d} K_{L_u - 1}(\lambda_u). \tag{2.2}$$

Also, on the "cube" $[-\pi, \pi]^d$, let m denote "normalized Lebesque measure", $m :=$ Lebesque measure$/(2\pi)^d$.

Lemma 2.3. *Suppose d is a positive integer. Suppose $f : [-\pi, \pi]^d \to \mathbb{C}$ is a continuous function. Then*

$$\int_{\vec{\lambda} \in [-\pi,\pi]^d} G_L(\vec{\lambda}) \cdot f(\vec{\lambda}) dm(\vec{\lambda}) \to f(\vec{0}) \text{ as } \min\{L_1, L_2, \ldots, L_d\} \to \infty.$$

Let us mention that Lemma 2.3 is a special case of the multivariate Fejér theorem, where the function f is a periodic function with period 2π in every coordinate. For a proof of the one dimensional case, see Rudin (1976).

Further notations will be introduced and used throughout the entire paper.
If $a_n \in (0, \infty)$ and $b_n \in (0, \infty)$ for all $n \in \mathbb{N}$ sufficiently large, the notation $a_n \ll b_n$ means that $\limsup_{n\to\infty} a_n/b_n < \infty$.
If $a_n \in (0, \infty)$ and $b_n \in (0, \infty)$ for all $n \in \mathbb{N}$ sufficiently large, the notation $a_n \lesssim b_n$ means that $\limsup_{n\to\infty} a_n/b_n \le 1$.
If $a_n \in (0, \infty)$ and $b_n \in (0, \infty)$ for all $n \in \mathbb{N}$ sufficiently large, the notation $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$.

3. Central Limit Theorems

In this section we introduce two limit theorems that help us build up the main result, presented also in this section, as Theorem 3.3.

Theorem 3.1. *Suppose d is a positive integer. Suppose also that $X := \left(X_k, k \in \mathbb{Z}^d\right)$ is a strictly stationary ρ'-mixing random field with the random variables X_k being real-valued such that $EX_0 = 0$ and $EX_0^2 < \infty$.*
Then the following two statements hold:
(I) The quantity

$$\sigma^2 := \lim_{\min\{L_1,L_2,\ldots,L_d\}\to\infty} \frac{ES^2(X,L)}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} \text{ exists in } [0,\infty), \text{ and}$$

(II) As $\min\{L_1, L_2, \ldots, L_d\} \to \infty$, $(L_1 \cdot L_2 \cdot \ldots \cdot L_d)^{-1/2} S(X, L) \Rightarrow N(0, \sigma^2)$. (Here and throughout the paper $\Rightarrow$ denotes convergence in distribution.)

Proof: The proof of the theorem has resemblance to arguments in earlier papers involving the ρ^*-mixing condition and similar properties as Theorem 3.1 (see Bradley, 1992 and Miller, 1994). The proof will be written out for the case $d \ge 2$ since it is essentially the same for the case $d = 1$, but the notations for the general case $d \ge 2$ are more complicated.

Proof of (I). Our task is to show that there exists a number $\sigma^2 \in [0, \infty)$ such that

$$\lim_{\min\{L_1,L_2,\ldots,L_d\}\to\infty} \frac{ES^2(X,L)}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} = \sigma^2. \tag{3.1}$$

For a given strictly stationary random field $X := \left(X_k, k \in \mathbb{Z}^d\right)$ with mean zero and finite second moments, if $\rho'(n) \to 0$ as $n \to \infty$ then $\zeta(n) \to 0$ as $n \to \infty$. Hence, by Bradley (2007) (Remark 29.4(V)(ii) and Remark 28.11(iii)(iv)), the random field X has exactly one continuous spectral density function, $\sigma^2 := f(1, 1, \ldots, 1)$, where $f : [-\pi, \pi]^d \to [0, \infty)$, and in addition, it is periodic with period 2π in every coordinate. In the following, by basic computations we compute the quantity given in (3.1). First we obtain that:

$$
\begin{aligned}
E\left|S\left(X,L\right)\right|^2 &= E\left|\sum_{k_1=1}^{L_1}\cdots\sum_{k_d=1}^{L_d}X_{(k_1,\ldots,k_d)}\right|^2 \\
&= \left(\sum_{k_1=1}^{L_1}\cdots\sum_{k_d=1}^{L_d}\right)\left(\sum_{l_1=1}^{L_1}\cdots\sum_{l_d=1}^{L_d}\right)EX_{(k_1,\ldots,k_d)}\overline{X_{(l_1,\ldots,l_d)}}.
\end{aligned}
\tag{3.2}
$$

We substitute the last term in the right-hand side of (3.2) by the following expression (see Bradley, 2007, Section 0.19):

$$
\begin{aligned}
&\frac{1}{(2\pi)^d}\left(\sum_{k_1=1}^{L_1}\cdots\sum_{k_d=1}^{L_d}\right)\left(\sum_{l_1=1}^{L_1}\cdots\sum_{l_d=1}^{L_d}\right)\int_{\lambda_1=-\pi}^{\pi}\cdots\int_{\lambda_d=-\pi}^{\pi} \\
&e^{i((k_1-l_1)\lambda_1+\ldots+(k_d-l_d)\lambda_d)}f(e^{i\lambda_1},\ldots,e^{i\lambda_d})d\lambda_d\ldots d\lambda_1 \\
&=\frac{1}{(2\pi)^d}\int_{\lambda_1=-\pi}^{\pi}\cdots\int_{\lambda_d=-\pi}^{\pi}f(e^{i\lambda_1},\ldots,e^{i\lambda_d})\cdot \\
&\cdot\left(\sum_{k_1=1}^{L_1}\sum_{l_1=1}^{L_1}e^{i(k_1-l_1)\lambda_1}\cdots\sum_{k_d=1}^{L_d}\sum_{l_d=1}^{L_d}e^{i(k_d-l_d)\lambda_d}\right)d\lambda_d\ldots d\lambda_1.
\end{aligned}
\tag{3.3}
$$

By (2.1), the right-hand side of (3.3) becomes:

$$
\begin{aligned}
&\frac{1}{(2\pi)^d}\int_{\lambda_1=-\pi}^{\pi}\cdots\int_{\lambda_d=-\pi}^{\pi}f(e^{i\lambda_1},\ldots,e^{i\lambda_d})\cdot \\
&\cdot\frac{\sin^2\left(L_1\lambda_1/2\right)}{\sin^2(\lambda_1/2)}\cdot\ldots\cdot\frac{\sin^2\left(L_d\lambda_d/2\right)}{\sin^2(\lambda_d/2)}d\lambda_d\ldots d\lambda_1 \\
&=\frac{1}{(2\pi)^d}\int_{\lambda_1=-\pi}^{\pi}\cdots\int_{\lambda_d=-\pi}^{\pi}f(e^{i\lambda_1},\ldots,e^{i\lambda_d})\cdot \\
&\cdot(L_1\cdot\ldots\cdot L_d)\cdot G_L(\lambda_1,\ldots,\lambda_d)d\lambda_d\ldots d\lambda_1,
\end{aligned}
\tag{3.4}
$$

therefore, by (3.2), (3.4) and the application of Lemma 2.3, we obtain that

$$
\begin{aligned}
&\lim_{\min\{L_1,\ldots,L_d\}\to\infty}\frac{ES^2\left(X,L\right)}{L_1\cdot\ldots\cdot L_d} \\
=&\lim_{\min\{L_1,\ldots,L_d\}\to\infty}\frac{1}{(2\pi)^d}\int_{\lambda_1=-\pi}^{\pi}\cdots\int_{\lambda_d=-\pi}^{\pi}G_L(\lambda_1,\ldots,\lambda_d)\cdot \\
&\cdot f(e^{i\lambda_1},\ldots,e^{i\lambda_d})d\lambda_d\ldots d\lambda_1 \\
=&f(1,\ldots,1).
\end{aligned}
$$

Hence, we can conclude that there exists a number $\sigma^2 := f(1,\ldots,1)$ in $[0,\infty)$ satisfying (3.1). This completes the proof of part (I).

Proof of (II). Refer now to Proposition 2.2 from Section 2. Let $u \in \{1,2,\ldots,d\}$ be arbitrary but fixed. Let $L^{(1)}, L^{(2)}, L^{(3)},\ldots$ be an arbitrary fixed sequence of elements of $\mathbb{N}^d$ such that for each $n \geq 1$, $L_u^{(n)} = n$ and $L_v^{(n)} \to \infty$ as $n \to \infty$, $\forall$ $v \in \{1,2,\ldots,d\}\setminus\{u\}$. It suffices to show that

$$
\frac{S\left(X,L^{(n)}\right)}{\sqrt{L_1^{(n)}\cdot L_2^{(n)}\cdot\ldots\cdot L_d^{(n)}}} \Rightarrow N(0,\sigma^2) \text{ as } n\to\infty. \tag{3.5}
$$

With no loss of generality, we can permute the indices in the coordinate system of $\mathbb{Z}^d$, in order to have $u = 1$, and as a consequence, we have:

$$L_1^{(n)} = n \text{ for } n \geq 1, \text{ and } L_v^{(n)} \to \infty \text{ as } n \to \infty, \ \forall \ v \in \{2, \ldots, d\}. \tag{3.6}$$

Thus for each $n \geq 1$, let us represent $L^{(n)} := \left(n, L_2^{(n)}, L_3^{(n)}, \ldots, L_d^{(n)}\right)$. We assume from now on, throughout the rest of the proof that $\sigma^2 > 0$. The case $\sigma^2 = 0$ holds trivially by an application of Chebyshev Inequality.

Step 1. A common technique used in proving central limit theorems for random fields satisfying strong mixing conditions is the truncation argument whose effect makes the partial sum of the bounded random variables converge weakly to a normal distribution while the tails are negligible. To achieve this, for each integer $n \geq 1$, define the (finite) positive number

$$c_n := \left(L_2^{(n)} \cdot L_3^{(n)} \cdot \ldots \cdot L_d^{(n)}\right)^{1/4}. \tag{3.7}$$

(3.6),

$$c_n \to \infty \text{ as } n \to \infty. \tag{3.8}$$

For each $n \geq 1$, we define the strictly stationary random field of bounded variables $X^{(n)} := \left(X_k^{(n)}, k \in \mathbb{Z}^d\right)$ as follows:

$$\forall\, k \in \mathbb{Z}^d, \ X_k^{(n)} := X_k I(|X_k| \leq c_n) - EX_0 I(|X_0| \leq c_n). \tag{3.9}$$

Hence, by simple computations we obtain that $\forall\, n \geq 1$,

$$EX_0^{(n)} = 0 \text{ and } VarX_0^{(n)} = E\left(X_0^{(n)}\right)^2 \leq EX_0^2 < \infty. \tag{3.10}$$

We easily also obtain that $\forall\, n \geq 1$,

$$\left|X_0^{(n)}\right| \leq 2c_n \text{ and } \left\|X_0^{(n)}\right\|_2 \leq \|X_0\|_2. \tag{3.11}$$

Next for $n \geq 1$, we define the strictly stationary random field of the tails of the X_k's, $k \in \mathbb{Z}^d$, $\widetilde{X}^{(n)} := \left(\widetilde{X}_k^{(n)}, k \in \mathbb{Z}^d\right)$ as follows (recall (3.9) and the assumption $EX_0 = 0$):

$$\forall\, k \in \mathbb{Z}^d, \ \widetilde{X}_k^{(n)} := X_k - X_k^{(n)} = X_k I(|X_k| > c_n) - EX_0 I(|X_0| > c_n). \tag{3.12}$$

As in (3.12), we similarly obtain by the dominated convergence theorem that

$$\forall\, n \geq 1, \ E\widetilde{X}_0^{(n)} = 0 \text{ and } E\left(\widetilde{X}_0^{(n)}\right)^2 \to 0 \text{ as } n \to \infty. \tag{3.13}$$

Note that $S\left(X, L^{(n)}\right) := \sum_k X_k = \sum_k X_k^{(n)} + \sum_k \widetilde{X}_k^{(n)}$, where all the sums are taken over all d-tuples $k := (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d$ such that $1 \leq k_u \leq L_u$ for all $u \in \{1, 2, \ldots, d\}$. Also, throughout the paper, unless specified, the notation $\sum_k$ will mean that the sum is taken over the same set of indices as above.

Step 2 (Parameters). For each $n \geq 1$, define the positive integer $q_n := [n^{1/4}]$, the greatest integer $\leq n^{1/4}$. Then it follows that

$$q_n \to \infty \text{ as } n \to \infty. \tag{3.14}$$

Recall that $\rho'(X, n) \to 0$ as $n \to \infty$. As a consequence, we have the following two properties:

$$\alpha(X, n) \to 0 \text{ as } n \to \infty, \text{ and also} \tag{3.15}$$

$$\text{there exists a positive integer j such that } \rho'(X, j) < 1. \tag{3.16}$$

Let such a j henceforth be fixed for the rest of the proof. By (3.15) and (3.14),

$$\alpha(X, q_n) \to 0 \text{ as } n \to \infty. \tag{3.17}$$

With $[x]$ denoting the greatest integer $\leq x$, define the positive integers $m_n,\ n \geq 1$ as follows:

$$m_n := \left[\min\left\{q_n, n^{1/10}, \alpha^{-1/5}(X, q_n)\right\}\right]. \tag{3.18}$$

By the equations (3.18), (3.14), and (3.17), we obtain the following properties:

$$m_n \to \infty \text{ as } n \to \infty, \tag{3.19}$$

$$m_n \leq q_n \text{ for all } n \geq 1, \tag{3.20}$$

$$\frac{m_n q_n}{n} \to 0 \text{ as } n \to \infty, \text{ and} \tag{3.21}$$

$$m_n \alpha(X, q_n) \to 0 \text{ as } n \to \infty. \tag{3.22}$$

For each $n \geq 1$, let p_n be the integer such that

$$m_n(p_n - 1 + q_n) < n \leq m_n(p_n + q_n). \tag{3.23}$$

Hence we also have that

$$p_n \to \infty \text{ as } n \to \infty \text{ and } m_n p_n \sim n. \tag{3.24}$$

Step 3 (The "Blocks"). In the following we decompose the partial sum of the bounded random variables $X_k^{(n)},\ k \in \mathbb{Z}^d$ into “big blocks” separated in between by “small blocks”. The “lengths” of both the big blocks and the small blocks, p_n and q_n respectively, have to “blow up” much faster than the (equal) numbers of big and small blocks, m_n (in addition to the fact that the “lengths of the “big blocks” need to “blow up” much faster than the “lengths” of the “small blocks”). This explains the way the positive integers $m_n,\ n \geq 1$ were defined in (3.18). Referring to the definition of the random variables $X_k^{(n)}$ in (3.9), for any $n \geq 1$ and any two positive integers $v \leq w$, define the random variable

$$Y^{(n)}(v, w) := \sum_k X_k^{(n)}, \tag{3.25}$$

where the sum is taken over all $k := (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d$ such that $v \leq k_1 \leq w$ and $1 \leq k_u \leq L_u^{(n)}$ for all $u \in \{2, \ldots, d\}$. Notice that for each $n \geq 1$, $S\left(X^{(n)}, L^{(n)}\right) = Y^{(n)}(1, n)$. Referring to (3.25), for each $n \geq 1$, define the random variables $U_k^{(n)}$ and $V_k^{(n)}$, as follows:

$$\forall\, k \in \{1, 2, \ldots, m_n\},\ U_k^{(n)} := Y^{(n)}\left((k-1)(p_n + q_n) + 1, kp_n + (k-1)q_n\right); \tag{3.26}$$
(“big blocks”)

$$\forall\, k \in \{1, 2, \ldots, m_n - 1\},\ V_k^{(n)} := Y^{(n)}(kp_n + (k-1)q_n + 1, k(p_n + q_n)); \tag{3.27}$$

("small blocks"), and

$$V_{m_n}^{(n)} := Y^{(n)}(m_n p_n + (m_n - 1)q_n + 1, n). \tag{3.28}$$

Note that by (3.20) and the first inequality in (3.23), for $n \geq 1$,

$$m_n p_n + (m_n - 1)q_n + 1 \leq m_n p_n + m_n q_n - m_n + 1 \leq n.$$

By (3.25), (3.26), (3.27), and (3.28),

$$\forall\, n \geq 1,\; S\left(X^{(n)}, L^{(n)}\right) = \sum_{k=1}^{m_n} U_k^{(n)} + \sum_{k=1}^{m_n} V_k^{(n)}. \tag{3.29}$$

Step 4 (Negligibility of the ”small blocks”). Note that by (3.27) and (3.28), $\sum_{k=1}^{m_n} V_k^{(n)}$ is the sum of at most $m_n \cdot q_n \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}$ of the random variables $X_k^{(n)}$. Therefore, by (3.16) and Bradley (2007), Theorem 28.10(I), for any $n \geq 1$, the following holds:

$$E\left|\sum_{k=1}^{m_n} V_k^{(n)}\right|^2 \leq C\left(m_n \cdot q_n \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}\right) E\left(X_0^{(n)}\right)^2, \tag{3.30}$$

where $C := j^d \left(1+\rho'(X,j)\right)^d / \left(1-\rho'(X,j)\right)^d$, and as a consequence, by (3.21) and (3.10), we obtain that

$$E\left|\frac{\sum_{k=1}^{m_n} V_k^{(n)}}{\sigma\sqrt{n \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}}}\right|^2 \leq \frac{C(m_n q_n) E\left(X_0^{(n)}\right)^2}{n \cdot \sigma^2} \to 0 \text{ as } n \to \infty. \tag{3.31}$$

Hence, the “small blocks” are negligible:

$$\frac{\sum_{k=1}^{m_n} V_k^{(n)}}{\sigma\sqrt{n \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}}} \to 0 \text{ in probability as } n \to \infty. \tag{3.32}$$

By an obvious analog of (3.31), followed by (3.13), for each $n \geq 1$, we obtain that

$$\frac{\sum_k \widetilde{X}_k^{(n)}}{\sigma\sqrt{n \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}}} \to 0 \text{ in probability as } n \to \infty. \tag{3.33}$$

Step 5 (Application of the Lyapounov CLT). For a given $n \geq 1$, by the definition of $U_k^{(n)}$ in (3.26) and the strict stationarity of the random field $X^{(n)}$, the random variables $U_1^{(n)}, U_2^{(n)}, \ldots, U_{m_n}^{(n)}$ are identically distributed. For each $n \geq 1$, let $\widetilde{U}_1^{(n)}, \widetilde{U}_2^{(n)}, \ldots, \widetilde{U}_{m_n}^{(n)}$ be independent, identically distributed random variables whose common distribution is the same as that of $U_1^{(n)}$. Hence, since $\forall\, n \geq 1, EX_0^{(n)} = 0$, we have the following:

$$E\widetilde{U}_1^{(n)} = EU_1^{(n)} = 0 \text{ and } Var\left(\sum_{k=1}^{m_n} \widetilde{U}_k^{(n)}\right) = m_n E\left(\widetilde{U}_1^{(n)}\right)^2 = m_n E\left(U_1^{(n)}\right)^2.$$

By (3.16), we can refer to Bradley (2007), Theorem 29.30, a result which gives us a Rosenthal inequality for ρ'-mixing random fields. Also, using the fact that $EU_1^2 \sim \sigma^2\left(p_n \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}\right)$ (see (3.1)), together with the equations (3.11), (3.10), and assuming without loss of generality that $EX_0^2 \le 1$, the following holds:

$$
\begin{aligned}
\frac{E\left(U_1^{(n)}\right)^4}{m_n\left(EU_1^2\right)^2} &\lesssim \frac{C_R\left(p_n \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)} \cdot E\left|X_0^{(n)}\right|^4 + \left(p_n \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)} \cdot EX_0^2\right)^2\right)}{m_n p_n^2 \sigma^4\left(L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}\right)^2} \\
&\le \frac{16 C_R p_n c_n^4\left(L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}\right)}{m_n p_n^2\left(L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}\right)^2 \sigma^4} + \frac{C_R p_n^2\left(L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}\right)^2}{m_n p_n^2\left(L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}\right)^2 \sigma^4} \\
&\le \frac{16 C_R}{m_n p_n \sigma^4} + \frac{C_R}{m_n \sigma^4} \to 0 \text{ as } n \to \infty \text{ by (3.24) and (3.19).}
\end{aligned}
\tag{3.34}
$$

Since $U_1 - U_1^{(n)}$ is the sum of $p_n \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}$ random variables $\widetilde{X}_k^{(n)}$, applying an obvious analog of (3.30), followed by (3.1) and (3.13), we have that as $n \to \infty$,

$$
\frac{E\left(U_1 - U_1^{(n)}\right)^2}{EU_1^2} \lesssim \frac{C p_n\left(L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}\right) E\left(\widetilde{X}_0^{(n)}\right)^2}{p_n\left(L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}\right)\sigma^2} = \frac{CE\left(\widetilde{X}_0^{(n)}\right)^2}{\sigma^2} \to 0.
$$

As a consequence, after an application of Minkowski Inequality to the quantity $\left|\|U_1\|_2 - \left\|U_1^{(n)}\right\|_2\right| / \|U_1\|_2$, we have that

$$
E\left(U_1^{(n)}\right)^2 \sim EU_1^2. \tag{3.35}
$$

Hence, by (3.34) and (3.35), the following holds:

$$
\frac{E\left(U_1^{(n)}\right)^4}{m_n\left(E\left(U_1^{(n)}\right)^2\right)^2} \sim \frac{E\left(U_1^{(n)}\right)^4}{m_n(EU_1^2)^2} \to 0 \text{ as } n \to \infty.
$$

Therefore, due to Lyapounov CLT (see Billingsley, 1995, Theorem 27.3), it follows that

$$
\left(\sqrt{m_n}\left\|U_1^{(n)}\right\|_2\right)^{-1} \sum_{k=1}^{m_n} \widetilde{U}_k^{(n)} \Rightarrow N(0,1) \text{ as } n \to \infty. \tag{3.36}
$$

Step 6. As in Bradley (2007), Theorem 29.32, we similarly obtain by (3.25), (3.26) and (3.22) that as $n \to \infty$,

$$
\sum_{k=1}^{m_n-1} \alpha\left(\sigma\left(U_j^{(n)}, 1 \le j \le k\right), \sigma\left(U_{k+1}^{(n)}\right)\right) \le \sum_{k=1}^{m_n-1} \alpha\left(X^{(n)}, q_n\right) \le m_n \alpha(X, q_n) \to 0.
$$

Hence, by (3.36) and by Bradley (2007), Theorem 25.56, the following holds:

$$
\left(\sum_{k=1}^{m_n} U_k^{(n)}\right) / \left(\sqrt{m_n}\left\|U_1^{(n)}\right\|_2\right) \Rightarrow N(0,1) \text{ as } n \to \infty. \tag{3.37}
$$

Refer to the first sentence of Step 5. For each $n \ge 1$,

$$
E\left(\sum_{k=1}^{m_n} U_k^{(n)}\right)^2 = m_n E\left(U_1^{(n)}\right)^2 + 2\sum_{k=1}^{m_n-1}\sum_{j=k+1}^{m_n} EU_k^{(n)} U_j^{(n)}. \tag{3.38}
$$

Using similar arguments as in Bradley (2007), Theorem 29.31 (Step 9), followed by (3.34) and (3.35), and (3.24), $E\left(U_1^{(n)}\right)^4/\left(E\left(U_1^{(n)}\right)^2\right)^2 \to C_R/\sigma^4$ as $n\to\infty$. Hence we obtain that $\left\|U_1^{(n)}\right\|_4^2 \ll E\left(U_1^{(n)}\right)^2$. As a consequence, by (3.38),

$$\left\|\sum_{k=1}^{m_n} U_k^{(n)}\right\|^2 \sim \left(m_n E\left(U_1^{(n)}\right)^2\right)^{1/2}. \tag{3.39}$$

Applying an obvious analog of (3.30) for

$$S\left(\widetilde{X}^{(n)}, L^{(n)}\right) := S\left(X, L^{(n)}\right) - S\left(X^{(n)}, L^{(n)}\right),$$

followed by (3.1) and (3.13), the following holds:

$$E\left(S\left(\widetilde{X}^{(n)}, L^{(n)}\right)\right)^2/E\left(S\left(X, L^{(n)}\right)\right)^2 \lesssim CE\left(\widetilde{X}_0^{(n)}\right)^2/\sigma^2 \to 0 \text{ as } n\to\infty. \tag{3.40}$$

Using Minkowski Inequality for

$$\left|\left\|S\left(X, L^{(n)}\right)\right\|_2 - \left\|S\left(X^{(n)}, L^{(n)}\right)\right\|_2\right| / \left\|S\left(X, L^{(n)}\right)\right\|_2,$$

by (3.40) it follows that

$$\left\|S\left(X^{(n)}, L^{(n)}\right)\right\|_2 \sim \left\|S\left(X, L^{(n)}\right)\right\|_2. \tag{3.41}$$

Now apply again Minkowski Inequality for

$$\left|\left\|\sum_{k=1}^{m_n} U_k^{(n)}\right\|_2 - \left\|S\left(X^{(n)}, L^{(n)}\right)\right\|_2\right| / \left\|S\left(X^{(n)}, L^{(n)}\right)\right\|_2,$$

and by the formulation of $S\left(X^{(n)}, L^{(n)}\right)$ given in (3.29), followed by (3.30), (3.39), (3.1) and by (3.21), we obtain that

$$\left\|S\left(X^{(n)}, L^{(n)}\right)\right\|_2 \sim \left\|\sum_{k=1}^{m_n} U_k^{(n)}\right\|_2. \tag{3.42}$$

Hence, by (3.39) and (3.41),

$$\left\|S\left(X, L^{(n)}\right)\right\|_2 \sim \left(m_n E\left(U_1^{(n)}\right)^2\right)^{1/2}.$$

As a consequence, by (3.37) and the fact that

$$\left\|S\left(X, L^{(n)}\right)\right\|_2 \sim \sigma\sqrt{n\cdot L_2^{(n)}\cdot\ldots\cdot L_d^{(n)}}$$

(see (3.1)), it follows the following:

$$\frac{\sum_{k=1}^{m_n} U_k^{(n)}}{\sigma\sqrt{n\cdot L_2^{(n)}\cdot\ldots\cdot L_d^{(n)}}} \Rightarrow N(0,1) \text{ as } n\to\infty. \tag{3.43}$$

Step 7. Refer to the definition of $S\left(X^{(n)}, L^{(n)}\right)$ given in (3.29). By (3.32) and (3.43), followed by Bradley (2007), Theorem 0.6, we obtain the following weak convergence:

$$\frac{S\left(X^{(n)}, L^{(n)}\right)}{\sigma\sqrt{n \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}}} \Rightarrow N(0,1) \text{ as } n \to \infty. \tag{3.44}$$

Refer now to the definition of $S\left(X, L^{(n)}\right)$ given just after (3.13). By another application of Theorem 0.6 from Bradley (2007) for (3.33) and (3.44), we obtain that (3.5) holds, and hence, the proof of (II) is complete. Moreover, the proof of the theorem is complete. □

Theorem 3.2. *Suppose d and m are each a positive integer. Suppose $X := (X_k, k \in \mathbb{Z}^d)$ is a strictly stationary ρ'-mixing random field with $X_k := (X_{k1}, X_{k2}, \ldots, X_{km})$ being (for each k) an m-dimensional random vector such that $\forall\, i \in \{1, 2, \cdots, m\}$, X_{ki} is a real-valued random variable with $EX_{ki} = 0$ and $EX_{ki}^2 < \infty$.*

Then the following statements hold:
(I) For any $i \in \{1, 2, \ldots, m\}$, the quantity

$$\sigma_{ii} = \lim_{min\{L_1, L_2, \ldots, L_d\} \to \infty} \frac{ES_{L,i}^2}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} \text{ exists in } [0, \infty),$$

where for each $L \in \mathbb{N}^d$ and each $i \in \{1, 2, \ldots, m\}$,

$$S_{L,i} := \sum_k X_{ki}, \tag{3.45}$$

with the sum being taken over all $k := (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d$ such that $1 \le k_u \le L_u$ for all $u \in \{1, 2, \ldots, d\}$.
(II) Also, for any two distinct elements i, $j \in \{1, 2, \ldots, m\}$,

$$\gamma(i,j) = \lim_{min\{L_1, L_2, \ldots, L_d\} \to \infty} \frac{E(S_{L,i} - S_{L,j})^2}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} \text{ exists in } [0, \infty).$$

(III) Furthermore, as $\min\{L_1, L_2, \ldots, L_d\} \to \infty$,

$$\frac{S(X, L)}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}} \Rightarrow N(0_m, \Sigma), \text{ where}$$

$$\Sigma := (\sigma_{ij}, 1 \le i \le j \le m) \text{ is the } m \times m \text{ covariance matrix defined by} \tag{3.46}$$

$$\text{for } i \ne j,\ \sigma_{ij} = \frac{1}{2}(\sigma_{ii} + \sigma_{jj} - \gamma(i,j)), \tag{3.47}$$

with σ_{ii} and $\gamma(i,j)$ defined in part (I), respectively in part (II).
(The fact that the matrix Σ in (III) is symmetric and nonnegative definite (and can therefore be a covariance matrix), is part of the conclusion of (III).)

Proof: A distant resemblance to this theorem is a bivariate central limit theorem of Miller (1995). The proof of Theorem 3.2 will be divided in the following parts:
Proof of (I) and (II). Since σ_{ii}, respectively $\gamma(i,j)$ exist by Theorem 3.1(I), parts (I) and (II) hold.
Proof of (III). For the clarity of the proof, the strategy used to prove this part is the following:
(i) It will be shown that the matrix Σ defined in part (III) is symmetric and nonnegative definite.

(ii) One will then let $Y := (Y_1, Y_2, \dots, Y_m)$ be a centered normal random vector with covariance matrix Σ, and the task will be to show that

$$\frac{S(X, L)}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}} \Rightarrow Y \text{ as } \min\{L_1, L_2, \dots, L_d\} \to \infty. \tag{3.48}$$

(iii) To accomplish that, by the Cramer-Wold Device Theorem (see Billingsley, 1995, Theorem 29.4) it suffices to show that for an arbitrary $t \in \mathbb{R}^m$,

$$t \cdot \frac{S_L}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}} \Rightarrow t \cdot Y \text{ as } \min\{L_1, L_2, \dots, L_d\} \to \infty, \tag{3.49}$$

where "·" denotes the scalar product.

Let us first show (i). In order to achieve this task, let us introduce $\Sigma^{(L)} := \left(\sigma_{ij}^{(L)}, 1 \le i \le j \le m\right)$ to be the $m \times m$ covariance matrix defined by

$$\sigma_{ij}^{(L)} = ES_{L,i}S_{L,j} = \frac{1}{2}\left(ES_{L,i}^2 + ES_{L,j}^2 - E(S_{L,i} - S_{L,j})^2\right). \tag{3.50}$$

Note that $\sigma_{ii}^{(L)} = ES_{L,i}^2$ for $i \in \{1, 2, \dots, m\}$. Our main goal is to prove that

$$\lim_{\min\{L_1, L_2, \dots, L_d\} \to \infty} \frac{\Sigma^{(L)}}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} = \Sigma \text{ (defined in (3.46))}. \tag{3.51}$$

It actually suffices to show that

$$\lim_{\min\{L_1, L_2, \dots, L_d\} \to \infty} \frac{\sigma_{ij}^{(L)}}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} = \sigma_{ij}, \ \forall\ 1 \le i \le j \le m. \tag{3.52}$$

By the definition of $\sigma_{ij}^{(L)}$ given in (3.50), followed by the distribution of the limit (each of the limits exist by Theorem 3.2, parts (I) and (II)), the left-hand side of (3.52) becomes:

$$\begin{aligned}
&\frac{1}{2} \lim_{\min\{L_1, L_2, \dots, L_d\} \to \infty} \frac{1}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} \left(ES_{L,i}^2 + ES_{L,j}^2 - E\left(S_{L,i} - S_{L,j}\right)^2\right) \\
&= \frac{1}{2}\left(\sigma_{ii} + \sigma_{jj} - \gamma(i,j)\right) = \sigma_{ij}.
\end{aligned}$$

Let us recall that each of these limits exist by Theorem 3.2, parts (I) and (II). Hence, (3.52) holds. As a consequence, (3.51) also holds.

In the following, one should mention that since $\Sigma^{(L)}$ is the $m \times m$ covariance matrix of $S_{L,i}$, one has that $\Sigma^{(L)}$ is symmetric and nonnegative definite. That is, $\forall\, r := (r_1, r_2, \dots, r_m) \in \mathbb{R}^m$, $r\Sigma^{(L)}r' \ge 0$. Therefore, $\forall\, r \in \mathbb{R}^m$, $r(L_1 \cdot L_2 \cdot \ldots \cdot L_d)^{-1}\Sigma^{(L)}r' \ge 0$, and moreover,

$$\forall\, r \in \mathbb{R}^m,\ r\left(\lim_{\min\{L_1, L_2, \dots, L_d\} \to \infty} (L_1 \cdot L_2 \cdot \ldots \cdot L_d)^{-1}\Sigma^{(L)}\right) r' \ge 0.$$

By (3.51), we get that $\forall\, r \in \mathbb{R}^m$, $r\Sigma r' \ge 0$, and hence, Σ is also symmetric (trivially by (3.51)) and nonnegative definite. Hence, there exists a centered normal random vector $Y := (Y_1, Y_2, \dots, Y_m)$ whose covariance matrix is Σ, and therefore, the proof of (i) is complete.

(ii) Let us now take $Y := (Y_1, Y_2, \dots, Y_m)$ be a centered normal random vector with covariance matrix Σ, defined in (3.46). As we mentioned above, the task now is to show that (3.48) holds. In order to accomplish this task, by part (iii), one would need to show (3.49).

(iii) So, let $t := (t_1, t_2, \ldots, t_m)$ be an arbitrary fixed element of $\mathbb{R}^m$. We can notice now that

$$t \cdot S_L = \sum_{i=1}^{m} t_i S_{L,i}, \text{ where } S_{L,i} \text{ is defined in (3.45).} \tag{3.53}$$

We can also notice that $t \cdot X_1, t \cdot X_2, \ldots$ is a strictly stationary ρ'-mixing random sequence with real-valued random variables that satisfy $E\,(t \cdot X_1) = t \cdot EX_1 = t \cdot 0_m = 0$, and $E\,(t \cdot X_1)^2 < \infty$. For these random variables we can apply Theorem 3.1. Therefore, we obtain that as $\min\{L_1, L_2, \ldots, L_d\} \to \infty$,

$$t \cdot \frac{S_L}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}} \Rightarrow N(0, \sigma^2), \tag{3.54}$$

where

$$\sigma^2 := \lim_{\min\{L_1, L_2, \ldots, L_d\} \to \infty} \frac{E\,(t \cdot S_L)^2}{L_1 \cdot L_2 \cdot \ldots \cdot L_d}. \tag{3.55}$$

Moreover, by (3.53), (3.50), and (3.51), (3.55) becomes:

$$\begin{aligned}
\sigma^2 &= \lim_{\min\{L_1, L_2, \ldots, L_d\} \to \infty} \frac{E\left(\sum_{i=1}^{m} t_i S_{L,i}\right)^2}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} \\
&= \lim_{\min\{L_1, L_2, \ldots, L_d\} \to \infty} \frac{1}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} \left(\sum_{i=1}^{m} t_i^2 E S_{L,i}^2 + \right. \\
&\left. + \sum_{1 \le i < j \le m} t_i t_j \left(E S_{L,i}^2 + E S_{L,j}^2 - E\,(S_{L,i} - S_{L,j})^2 \right) \right) \\
&= t \left(\lim_{\min\{L_1, L_2, \ldots, L_d\} \to \infty} \frac{\Sigma^{(L)}}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} \right) t' = t \Sigma t'.
\end{aligned} \tag{3.56}$$

By (3.54) and (3.56), one can conclude that

$$t \cdot \frac{S_L}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}} \Rightarrow N\left(0, t\Sigma t'\right) \text{ as } \min\{L_1, L_2, \ldots, L_d\} \to \infty. \tag{3.57}$$

Also, since the random vector Y is centered normal with covariance matrix Σ, one has that $t \cdot Y$ is a normal random variable with mean 0 and variance (1×1 covariance matrix) $t\Sigma t'$. Hence, by (3.57), (3.49) holds, therefore (3.48) holds. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Suppose H is a separable real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_H$. Suppose $X := (X_k, k \in \mathbb{Z}^d)$ is a strictly stationary ρ'-mixing random field with the random variables X_k being H-valued, such that*

$$EX_0 = 0_H \text{ and} \tag{3.58}$$

$$E\,\|X_0\|_H^2 < \infty. \tag{3.59}$$

Suppose $\{e_i\}_{i \ge 1}$ is an orthonormal basis of H and that $X_{ki} := \langle X_k, e_i \rangle$ for each pair (k, i).

Then the following statements hold:

(I) For each $i \in \mathbb{N}$, the quantity

$$\sigma_{ii} = \lim_{min\{L_1, L_2, \ldots, L_d\} \to \infty} \frac{E S_{L,i}^2}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} \text{ exists in } [0, \infty), \text{ where}$$

$$S_{L,i} := \sum_{k} X_{ki}, \text{ the sum being taken over all } k := (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d \tag{3.60}$$

such that $1 \le k_u \le L_u$ *for all* $u \in \{1, 2, \ldots, d\}$.
(II) Also, for any two distinct elements, $i, j \in \mathbb{N}$,

$$\gamma(i,j) = \lim_{\min\{L_1, L_2, \ldots, L_d\} \to \infty} \frac{E(S_{L,i} - S_{L,j})^2}{L_1 \cdot L_2 \cdot \ldots \cdot L_d} \text{ exists in } [0, \infty).$$

(III) Furthermore, as $\min\{L_1, L_2, \ldots, L_d\} \to \infty$,

$$\frac{S(X,L)}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}} \Rightarrow N\left(0_H, \Sigma^{(\infty)}\right),$$

where the "covariance operator" $\Sigma^{(\infty)} := (\sigma_{ij}, i \ge 1, j \ge 1)$ *is symmetric, nonnegative definite, has finite trace and it is defined by*

$$\text{for } i \neq j, \ \sigma_{ij} = \frac{1}{2}(\sigma_{ii} + \sigma_{jj} - \gamma(i,j)), \tag{3.61}$$

with σ_{ii} *and* $\gamma(i,j)$ *defined in part (I), respectively in part (II). (Recall that* $\Rightarrow$ *denotes convergence in distribution and also the statement before Lemma 2.1.)*

Proof: The proof of the theorem will be divided in the following parts:
Proof of (I) and (II). Since σ_{ii}, respectively $\gamma(i,j)$ exist by Theorem 3.1(I), parts (I) and (II) hold.
Proof of (III). The rest of the proof will be divided into five short steps, as follows:
Step 1. Since the Hilbert space H is separable, one can consider working with the separable Hilbert space l_2. Let us recall that $\forall\, k \in \mathbb{Z}^d$, $X_k = (X_{k1}, X_{k2}, X_{k3}, \cdots)$ is an l_2-valued random variable with real-valued components such that

$$EX_{ki} = 0, \ \forall\, i \ge 1 \text{ and} \tag{3.62}$$

$$E\|X_k\|_H^2 < \infty. \tag{3.63}$$

For any given $m \in \mathbb{N}$, if one considers the first m coordinates of the l_2-valued random variable X_k, $X_k^{(m)} := (X_{k1}, X_{k2}, \ldots, X_{km})$, by Theorem 3.2 we obtain:

$$\frac{S_L^{(m)}}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}} \Rightarrow N\left(0_m, \Sigma^{(m)}\right) \text{ as } \min\{L_1, L_2, \ldots, L_d\} \to \infty, \tag{3.64}$$

where $\Sigma^{(m)} := (\sigma_{ij}, 1 \le i \le j \le m)$ is the $m \times m$ covariance matrix defined as in (3.46). Let us specify that here and below, for any given $L \in \mathbb{N}^d$ and $m \in \mathbb{N}$, the random variable $S_L^{(m)}$ is defined by:

$$S_L^{(m)} := \sum_{k} X_k^{(m)}, \text{ the sum being taken over all } k := (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d$$

such that $1 \le k_u \le L_u$ for all $u \in \{1, 2, \ldots, d\}$.

Step 2. Suppose $m \in \mathbb{N}$. Let $\widetilde{Y}^{(m)} := \left(Y_1^{(m)}, Y_2^{(m)}, \ldots, Y_m^{(m)}\right)$ be an $\mathbb{R}^m$-valued random vector whose distribution on $(\mathbb{R}^m, \mathcal{R}^m)$ is $N\left(0_m, \Sigma^{(m)}\right), \Sigma^{(m)}$ being the same covariance matrix defined in (3.46). By Step 1, we have that

$$\frac{S_L^{(m)}}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}} \Rightarrow \widetilde{Y}^{(m)} \text{ as } \min\{L_1, L_2, \ldots, L_d\} \to \infty. \tag{3.65}$$

Let μ_m be the probability measure on $(\mathbb{R}^m, \mathcal{R}^m)$ of the random vector $\widetilde{Y}^{(m)}$ and let μ_{m+1} be the probability measure on $(\mathbb{R}^{m+1}, \mathcal{R}^{m+1})$ of the random vector $\widetilde{Y}^{(m+1)} := \left(Y_1^{(m+1)}, Y_2^{(m+1)}, \ldots, Y_m^{(m+1)}, Y_{m+1}^{(m+1)}\right)$, whose distribution is $N\left(0_{m+1}, \Sigma^{(m+1)}\right)$. One should specify that $\Sigma^{(m+1)} := (\sigma_{ij}, 1 \le i \le j \le m+1)$ is the $(m+1) \times (m+1)$ covariance matrix defined in (3.46), where the integer m in (3.46) corresponds to $m+1$ here.

Claim 3.1. *For each $m \in \mathbb{N}$, $\left(Y_1^{(m+1)}, Y_2^{(m+1)}, \ldots, Y_m^{(m+1)}\right)$ (that is, the first m coordinates of the random vector $\widetilde{Y}^{(m+1)}$) has the same distribution as $\widetilde{Y}^{(m)} := \left(Y_1^{(m)}, Y_2^{(m)}, \ldots, Y_m^{(m)}\right)$.*

Proof: Since the random vector $\tilde{Y}^{(m+1)}$ is (multivariate) centered normal, it follows automatically that $\left(Y_1^{(m+1)}, Y_2^{(m+1)}, \ldots, Y_m^{(m+1)}\right)$ (the first m coordinates) is centered normal. For the two centered normal random vectors $\widetilde{Y}^{(m)}$ and see above $\left(Y_1^{(m+1)}, Y_2^{(m+1)}, \ldots, Y_m^{(m+1)}\right)$, the $m \times m$ covariance matrices are the same (with the common entries being the elements σ_{ii} and σ_{ij} defined in Theorem 3.2). From this observation, as well as the fact that a (multivariate) centered normal distribution is uniquely determined by its covariance matrix, Claim 3.1 follows. □

Now, by Kolmogorov's Existence Theorem (see Billingsley, 1995, Theorem 36.2), there exists on some probability space $(\Omega, \mathcal{F}, P)$ a sequence of random variables $Y := (Y_1, Y_2, Y_3, \ldots)$ such that for each $m \ge 1$, the m-dimensional random vector $(Y_1, Y_2, \ldots, Y_m)$ has distribution μ_m on $(\mathbb{R}^m, \mathcal{R}^m)$.

Claim 3.2. *Y is a centered normal l_2-valued random variable.*

Proof: First of all, one should prove that Y is an l_2-valued random variable, whose (random) norm has a finite second moment; that is,

$$E\|Y\|_{l_2}^2 < \infty. \tag{3.66}$$

More precisely, one should check that

$$\sum_{i=1}^{\infty} EY_i^2 = \sum_{i=1}^{\infty} \sigma_{ii} < \infty, \text{ where } \sigma_{ii} = Cov(Y_i, Y_i) = EY_i^2. \tag{3.67}$$

Since for every $i \ge 1$, $S_{L,i}$ is the sum of $L_1 \cdot L_2 \cdot \ldots \cdot L_d$ real-valued random variables X_{ki}, by an obvious analog of (3.30), followed by the definition of σ_{ii}, given in part (I) of the theorem, we obtain the following inequality:

$$\sigma_{ii} \le C \cdot E|X_{0i}|^2, \text{ where } C \text{ is the constant defined just after (3.30)} \tag{3.68}$$

(with $j \ge 1$ fixed such that $\rho'(X, j) < 1$). Therefore, by (3.68) and (3.63),

$$\sum_{i=1}^{\infty} \sigma_{ii} \le C \sum_{i=1}^{\infty} E|X_{0i}|^2 < \infty.$$

Hence, (3.67) holds, that is Y is an l_2-valued random variable, whose (random) norm has a finite second moment. In order to prove that Y is a *normal* l_2-valued random variable, it now suffices to show the following:

$$\forall\, m \geq 1 \text{ and } \forall\,(r_1, r_2, \ldots, r_m) \in \mathbb{R}^m, \text{ the real-valued random variable} \\ \sum_{i=1}^{m} r_i Y_i \text{ is normal (possibly degenerate).} \tag{3.69}$$

In order to show (3.69), let $m \geq 1$ and $(r_1, r_2, \ldots, r_m) \in \mathbb{R}^m$. As we mentioned earlier, for each $m \geq 1$, the random vector $(Y_1, Y_2, \ldots, Y_m)$ is centered normal with covariance matrix $\Sigma^{(m)}$, defined in (3.46). Therefore, $\sum_{i=1}^{m} r_i Y_i$ is a centered normal real random variable. Hence, Y is a centered normal l_2-valued random variable (possibly degenerate) whose "covariance operator" is defined in (3.61), and therefore, the proof of Claim 3.2 is complete. □

Step 3. Refer now to Proposition 2.2 from Section 2. Let $u \in \{1, 2, \ldots, d\}$ be arbitrary but fixed. Let $L^{(1)}, L^{(2)}, L^{(3)}, \ldots$ be an arbitrary fixed sequence of elements of $\mathbb{N}^d$ such that for each $n \geq 1$, $L_u^{(n)} = n$ and $L_v^{(n)} \to \infty$ as $n \to \infty$, $\forall\, v \in \{1, 2, \ldots, d\} \setminus \{u\}$.

Suppose $m \geq 1$. Consider the following sequence:

$$\frac{S^{(m)}\left(X, L^{(1)}\right)}{\sqrt{L_1^{(1)} \cdot L_2^{(1)} \cdot \ldots \cdot L_d^{(1)}}}, \frac{S^{(m)}\left(X, L^{(2)}\right)}{\sqrt{L_1^{(2)} \cdot L_2^{(2)} \cdot \ldots \cdot L_d^{(2)}}}, \ldots, \frac{S^{(m)}\left(X, L^{(n)}\right)}{\sqrt{L_1^{(n)} \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}}}, \ldots.$$

By Step 1, one has the following:

$$\frac{S^{(m)}\left(X, L^{(n)}\right)}{\sqrt{L_1^{(n)} \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}}} \Rightarrow N\left(0_m, \Sigma^{(m)}\right) \text{ as } n \to \infty, \tag{3.70}$$

where $\Sigma^{(m)}$ is the $m \times m$ covariance matrix defined in (3.46).

Step 4. Let $\mathcal{P}$ denote the family of distributions of the l_2-valued random variables $S_L/\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}$, $L \in \mathbb{N}^d$. By Lemma 2.1, in order to show that $\mathcal{P}$ is tight, one should show that

$$\lim_{N\to\infty} \sup_{L\in\mathbb{N}^d} E\left(\sum_{i=N}^{\infty}\left\langle \frac{S_L}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}}, e_i\right\rangle^2\right) = 0, \tag{3.71}$$

as well as the fact that for $N = 1$ the supremum in (3.71) is finite.

Let $N \geq 1$ and $L \in \mathbb{N}^d$. Then using (3.60), followed by an obvious analog of (3.30), we obtain the following:

$$E\left(\sum_{i=N}^{\infty}\left\langle \frac{S_L}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}}, e_i\right\rangle^2\right) = \frac{1}{L_1 \cdot L_2 \cdot \ldots \cdot L_d}\sum_{i=N}^{\infty} ES_{L,i}^2 \leq C \sum_{i=N}^{\infty} E|X_{0i}|^2.$$

Since $E\|X_0\|_H^2 < \infty$, one has that

$$\lim_{N\to\infty} \sum_{i=N}^{\infty} E|X_{0i}|^2 = 0. \tag{3.72}$$

Also by (3.59), for $N = 1$ the sum in (3.72) is finite. Hence (3.71) holds, and as a consequence, $\mathcal{P}$ is tight. Moreover, $\mathcal{P}$ is tight along the sequence $L^{(1)}, L^{(2)}, L^{(3)}, \cdots$, hence the family of distributions $\left\{S\left(X, L^{(n)}\right)/\sqrt{L_1^{(n)} \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}}\right\}$ is tight. As a consequence, the sequence $S\left(X, L^{(n)}\right)/\sqrt{L_1^{(n)} \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}}$ contains a weakly convergent subsequence.

Step 5. Let Q be an infinite set in $\mathbb{N}$. Assume that as $n \to \infty$, $n \in Q$, the sequence $S\left(X, L^{(n)}\right)/\sqrt{L_1^{(n)} \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}} \Rightarrow W := (W_1, W_2, W_3, \ldots)$.

By Step 3, $(W_1, W_2, \ldots, W_m)$ is $N\left(0_m, \Sigma^{(m)}\right)$, where $\Sigma^{(m)} := (\sigma_{ij}, 1 \le i \le j \le m)$ is the $m \times m$ covariance matrix defined in (3.46). Hence, the distribution of the random vector $(W_1, W_2, \ldots, W_m)$ is the same as the distribution of $Y^{(m)}, \forall\, m$. Thus the distributions of W and Y are identical. Therefore,

$$\frac{S\left(X, L^{(n)}\right)}{\sqrt{L_1^{(n)} \cdot L_2^{(n)} \cdot \ldots \cdot L_d^{(n)}}} \Rightarrow Y \text{ as } n \to \infty,\ n \in Q. \tag{3.73}$$

Hence, we obtain that the convergence in (3.73) holds along the entire sequence of positive integers, and as a consequence,

$$\frac{S(X, L)}{\sqrt{L_1 \cdot L_2 \cdot \ldots \cdot L_d}} \Rightarrow Y \text{ as } \min\{L_1, L_2, \ldots, L_d\}.$$

Therefore, part (III) holds, and hence, the proof of the theorem is complete. □

Acknowledgment

The result is part of the author's Ph.D. thesis at Indiana University (Bloomington). The author thanks her advisor, Professor Richard Bradley, to whom she is greatly indebted for his advice and support not only in this work but also during the graduate years at Indiana University.

References

P. Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, third edition (1995). ISBN 0-471-00710-2. A Wiley-Interscience Publication. MR1324786.

R. C. Bradley. On the spectral density and asymptotic normality of weakly dependent random fields. *J. Theoret. Probab.* **5** (2), 355–373 (1992). MR1157990.

R. C. Bradley. *Introduction to strong mixing conditions. Vol. 1, 2, 3*. Kendrick Press, Heber City, UT (2007). ISBN 0-9740427-6-5. MR2325294.

J. Dedecker and F. Merlevède. Necessary and sufficient conditions for the conditional central limit theorem. *Ann. Probab.* **30** (3), 1044–1081 (2002). MR1920101.

R. G. Laha and V. K. Rohatgi. *Probability theory*. John Wiley & Sons, New York-Chichester-Brisbane (1979). ISBN 0-471-03262-*. Wiley Series in Probability and Mathematical Statistics. MR534143.

V. V. Mal′tsev and E. I. Ostrovskiĭ. The central limit theorem for stationary processes in Hilbert space. *Teor. Veroyatnost. i Primenen.* **27** (2), 337–339 (1982). MR657927.

F. Merlevède. On the central limit theorem and its weak invariance principle for strongly mixing sequences with values in a Hilbert space via martingale approximation. *J. Theoret. Probab.* **16** (3), 625–653 (2003). MR2009196.

F. Merlevède, M. Peligrad and S. Utev. Sharp conditions for the CLT of linear processes in a Hilbert space. *J. Theoret. Probab.* **10** (3), 681–693 (1997). MR1468399.

C. Miller. Three theorems on ρ^*-mixing random fields. *J. Theoret. Probab.* **7** (4), 867–882 (1994). MR1295544.

C. Miller. A central limit theorem for the periodograms of a ρ^*-mixing random field. *Stochastic Process. Appl.* **60** (4), 313–330 (1995).

W. Rudin. *Principles of mathematical analysis.* Third edition. McGraw-Hill Book Co., New York (1976). MR0166310.