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Central Limit Theorem for Excited Random Walk in the Recurrent Regime

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Abstract. We consider excited random walk on a strip. We assume that the cookies are positive and that the total expected drift per site is less than $1/L$ where L is the width of the strip. We prove a quenched limit theorem claiming that the position of the walker converges after the diffusive rescaling to a perturbed Brownian Motion.

Let $\mathcal{Y} = \mathbb{Z} \times (\mathbb{Z}/L\mathbb{Z})$, where $L > 1$ is an integer, $G = \{-e_1, e_1, -e_2, e_2\}$ where e_j are coordinate vectors. We denote the coordinates of points $y \in \mathcal{Y}$ by $(x(y), s(y))$. Consider a cookie environment on $\mathcal{Y}$, that is, for each $y \in \mathcal{Y}$, $j \in \mathbb{N}$, there is a probability distribution $\omega(y, j, e)$ on G. Consider an excited random walk $Y_n = (X_n, S_n)$ that is

$$\mathbb{P}(Y_{n+1} - Y_n = e | Y_1, \dots, Y_n) = \omega(Y_n, l_n, e)$$

where l_n is the number of visits to Y_n by time n. (We denote by $\mathbb{P}$ and $\mathbb{E}$ the quenched probability and expectation with fixed ω and by $\mathbf{P}$ and $\mathbf{E}$ the annealed probability and expectation.) Y_n is called (multi-)*excited random walk* (ERW). We make the following assumptions:

(A) $\delta(y, j) := \omega(y, j, e_1) - \omega(y, j, -e_1) \geq 0$,
(B) There exists $\kappa > 0$ such that $\omega(y, j, e) \geq \kappa$,
(C) ω is stationary with respect to G-shifts and ergodic.
(D) Let $\delta(y) = \sum_{j=1}^{\infty} \delta(y, j)$ then

$$\delta := \mathbf{E}(\delta(y)) < \frac{1}{L}.$$

(E) For each $\varepsilon > 0$ there exists $N(\varepsilon, y)$ such that for each $j \geq N$, for each $e \in G$ $|\omega(y, j, e) - \frac{1}{4}| < \varepsilon$. Moreover $\mathbf{E}(N(\varepsilon, y)) < \infty$.

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The quantity δ introduced in (D) plays a crucial role in description of the behavior of ERW. In particular Y_n is recurrent in the sense that every site is visited infinitely often iff $\delta L \leq 1$, see Zerner (2005, 2006); Aschenbrenner (2010). (In case $\delta L < 1$ which is a subject of the our work recurrence also follows from Lemma 8 of the present paper.) Several papers addressed the limiting behavior of the ERW in the transient regime Mountford et al. (2006); Basdevant and Singh (2008a,b); Kosygina and Zerner (2008); Kosygina and Mountfors (2011). Our paper deals with recurrent ERW.

Let $\mathcal{B}(t)$ denote the Brownian motion with variance $\frac{t}{2}$. Recall (Chaumont and Doney, 1999) that for all $\alpha, \beta < 1$ and for almost every realization of $\mathcal{B}$ there exists a unique solution $\mathcal{W}(t)$ of the equation

$$\mathcal{W}(t) = \mathcal{B}(t) + \alpha \max_{[0,t]} \mathcal{W}(s) + \beta \min_{[0,t]} \mathcal{W}(s) \tag{1}$$

which is called (α, β)-*perturbed Brownian Motion.*

Define $\mathcal{W}_n(t)$ by setting $\mathcal{W}_n(m/n) = \frac{X_m}{\sqrt{n}}$ and interpolating linearly in between.

Theorem 1. *For almost every ω, $\mathcal{W}_n$ converges weakly as $n \to \infty$ to (α, β)-perturbed Brownian Motion where $\alpha = -\beta = \delta L$.*

Remark 2. A similar result is valid for ERW on $\mathbb{Z}$ with obvious modifications. Namely, $G = \{-e, +e\}$, condition (E) becomes $|\omega(y, j, e) - \frac{1}{2}| < \varepsilon$ and the variance of the limiting Brownian Motion equals t.

Remark 3. Our result leaves open the critical case $\delta L = 1$. (Observe that (1) is not well posed if $\alpha = 1$.)

We divide the proof into several steps. Fix $\mathbf{T} > 0$.

Lemma 4. *For any m there is a constant γ_m^- such that for any ω, for any stopping time σ, for any numbers $R \in \mathbb{R}_+$, $N \in \mathbb{N}$ we have*

$$\mathbb{P}\left(\min_{k \leq N} (X_{\sigma+k} - X_\sigma) \leq -R\sqrt{N}\right) \leq \frac{\gamma_m^-}{R^{2m}}.$$

In particular

$$\mathbb{P}\left(\min_{[0,\mathbf{T}]} \mathcal{W}_n(t) < -R\right) \leq \frac{\hat{\gamma}_m^-}{R^{2m}}$$

where $\hat{\gamma}_m^- = \mathbf{T}^m \gamma_m^-$.

Proof: Denote

$$\Delta_k = X_{k+1} - X_k, \quad \bar{\Delta}_k = \mathbb{E}(\Delta_k | Y_1, \dots, Y_k) = \delta(Y_k, l_k),$$

$$C_n = \sum_{k=0}^{n-1} \bar{\Delta}_k, \quad B_n = \sum_{k=0}^{n-1} \left[\Delta_k - \bar{\Delta}_k\right].$$

By assumption (A), $X_{\sigma+k} - X_\sigma \geq B_{\sigma+k} - B_\sigma$. Since $M_k = B_{\sigma+k} - B_\sigma$ is a martingale with respect to the σ-algebra generated by $\Delta_0, \dots, \Delta_{\sigma+k-1}$ and the quadratic variation of M grows at most linearly, it follows from Hall and Heyde (1980), Theorem 2.11 that that for each $m \in \mathbb{N}$ there is a constant γ_m^- such that

$$\mathbb{E}((\max_{k \leq n} |M_k|)^m) \leq \gamma_m^- n^m$$

and so by Markov inequality

$$\mathbb{P}(\max_{k\le n}|M_k| \ge R\sqrt{n}) \le \frac{\gamma_m^-}{R^{2m}}. \tag{2}$$

which implies the result we need. $\square$

Denote

$$A_{n_0} = \left\{ \omega : \sum_{x(y)=-\frac{(1-\delta L)n}{3}}^{n} \delta(y) < \frac{(2+\delta L)n}{3} \text{ for all } n \ge n_0 \right\}.$$

Note that by the Ergodic Theorem

$$\mathbf{P}(A_{n_0}) \to 1 \text{ as } n_0 \to \infty. \tag{3}$$

Let T denote the space shift $(T^k\omega)((x,s),j,e) = \omega((x+k,s),j,e)$

Lemma 5. *There is a constant γ_m^+ such that for any $n_0 \in \mathbb{N}$, for any ω such that $T^x\omega \in A_{n_0}$ for any stopping time σ such that $X_\sigma = x$, for any numbers $R \in \mathbb{R}_+, N \in \mathbb{N}$ such that $R\sqrt{N} \ge n_0$ we have*

$$\mathbb{P}\left(\max_{k\le N}(X_{\sigma+k} - X_\sigma) \ge R\sqrt{N}\right) \le \frac{\gamma_m^+}{R^m}.$$

In particular for almost every ω we have

$$\mathbb{P}(\max_{[0,\mathbf{T}]} \mathcal{W}_n(t) > R) \le \frac{\hat{\gamma}_m^+}{R^m}$$

provided that n is large enough, where $\hat{\gamma}_m^+ = \mathbf{T}^m\gamma_m^+$.

Proof: Denote

$$\tilde{X}_k = X_{\min(\sigma+k,\tilde{\sigma})} - X_\sigma, \quad \tilde{M}_k = M_{\min(k,\tilde{\sigma}-\sigma)}$$

where M is the martingale from the proof of Lemma 4 and $\tilde{\sigma}$ is the first time after σ when $X_{\tilde{\sigma}} = X_\sigma - \left[R\sqrt{N}\frac{1-\delta L}{3}\right]$. In view of Lemma 4 it suffices to show that given m there is a constant $\bar{\gamma}_m$ such that

$$\mathbb{P}\left(\max \tilde{X}_k \ge R\sqrt{N}\right) \le \frac{\bar{\gamma}_m}{R^{2m}}.$$

By the definition of A_{n_0} we have $\tilde{X}_k \ge \tilde{M}_k + R\sqrt{N}\frac{2+\delta L}{3}$ so if $\tilde{X}_k \ge R\sqrt{N}$ then $\tilde{M}_k \ge R\sqrt{N}\frac{1-\delta L}{3}$. Now the statement of the lemma follows from (2). $\square$

Let $r_n = \max_{k\le n}(X_k) - \min_{k\le n}(X_k)$ denote the range of the walk. Define $\mathcal{B}_n(t)$ by setting $\mathcal{B}_n(\frac{m}{n}) = \frac{B_m}{\sqrt{n}}$ and interpolating linearly in between.

Lemma 6. *For almost every ω $\mathcal{B}_n$ converges weakly to $\mathcal{B}$ as $n \to \infty$.*

Proof: Since B_n is a martingale it suffices, due to Hall and Heyde (1980), Theorem 4.4, to show that for almost every ω

$$\sup_{t\in[0,\mathbf{T}]} \left| \frac{V_{[nt]}}{n} - \frac{t}{2} \right| \to 0 \text{ in probability as } n \to \infty$$

where V_n is the quadratic variation of B_n. For the discrete time process it is enough to show that for almost every ω

$$\max_{0\leq m\leq n}\left|\frac{V_m}{n}-\frac{m}{2n}\right|\to 0 \text{ in probability as } n\to\infty.$$

Fix $\varepsilon>0$. Choose N_0 such that

$$\mathbf{E}([N(\varepsilon,y)-N_0]^+)<\varepsilon \tag{4}$$

where $N(\varepsilon,y)$ is a constant from condition (E). Split $V_m=V_m^-+V_m^+$ where

$$V_m^-=\sum_{k=0}^{m-1}\mathbb{E}\left(\left[\Delta_k-\bar{\Delta}_k\right]^2|Y_1\dots Y_k\right)I(l_k\leq N_0),$$

$$V_m^+=\sum_{k=0}^{m-1}\mathbb{E}\left(\left[\Delta_k-\bar{\Delta}_k\right]^2|Y_1\dots Y_k\right)I(l_k> N_0).$$

Then $V_m^-\leq 4N_0Lr_m\ll n$ (by Lemmas 4 and 5) whereas

$$V_m^+=\frac{m}{2}+\epsilon_m'+\epsilon_m''$$

where

$$\epsilon_m'=\sum_k\left(\mathbb{E}\left(\left[\Delta_k-\bar{\Delta}_k\right]^2|Y_1\dots Y_k\right)-\frac{1}{2}\right)I(l_k>\max(N(\varepsilon,Y_k),N_0)),$$

$$\epsilon_m''=\sum_k\left(\mathbb{E}\left(\left[\Delta_k-\bar{\Delta}_k\right]^2|Y_1\dots Y_k\right)-\frac{1}{2}\right)I(N_0<l_k\leq N(\varepsilon,Y_k)).$$

Observe that on $l_k>N(\varepsilon,Y_k)$ we have

$$\left|\mathbb{E}\left(\left[\Delta_k-\bar{\Delta}_k\right]^2|Y_1\dots Y_k\right)-\frac{1}{2}\right|=$$

$$\left|\left[\omega(Y_k,l_k,e_1)+\omega(Y_k,l_k,-e_1)-\frac{1}{2}\right]-\left[\omega(Y_k,l_k,e_1)-\omega(Y_k,l_k,-e_1)\right]^2\right|\leq 2\varepsilon+(2\varepsilon)^2$$

and so $|\varepsilon_m''|\leq\left(2\varepsilon+(2\varepsilon)^2\right)n$. On the other hand

$$|\varepsilon_m''|\leq\sum^{*}[N(\varepsilon,y)-N_0]_+ \tag{5}$$

where the summation in (*) runs over y with

$$\min_{k\leq n}(X_k)\leq x(y)\leq\max_{k\leq n}(X_k).$$

So (4) and the ergodic theorem ensure that $|\varepsilon_m''|$ is less than $2\varepsilon Lr_n$ provided that r_n is large enough (if r_n is small then our claim that $|\varepsilon_m''|\ll n$ is obvious). This concludes the proof of Lemma 6. □

Lemma 7. *$\{\mathcal{W}_n\}$ is tight.*

Proof: Since $X_0=0$ Billingsley (1999), Lemma 8.3 implies that in order to prove tightness it suffices to show that for almost all ω given positive constants ε,η there exists a positive constant δ such that if n is sufficiently large then for all $t\leq\mathbf{T}$

$$\frac{1}{\delta}\mathbb{P}\left(\sup_{s\in[t,t+\delta]}|W_n(s)-W_n(t)|\geq\varepsilon\right)\leq\eta.$$

Without rescaling this amounts to showing that for all $n_1 \leq n\mathbf{T}$ we have

$$\frac{1}{\delta}\mathbb{P}\left(\max_{n_1 \leq n_2 \leq n_1+\delta n} |X_{n_2} - X_{n_1}| \geq \varepsilon\sqrt{n}\right) \leq \eta.$$

Take δ such that

$$\frac{\gamma_2^- \delta}{\left(\frac{23\varepsilon}{25}\right)^4} < \eta \text{ and } \frac{\gamma_2^+ \delta}{\left(\frac{23\varepsilon}{25}\right)^4} < \eta \tag{6}$$

By Lemmas 4 and 5 given η, δ there exists R such that

$$\mathbb{P}\left(\max_{k \leq \mathbf{T}n} |X_k| \geq R\sqrt{n}\right) \leq \frac{\delta\eta}{3}$$

so it suffices to show that

$$\frac{1}{\delta}\mathbb{P}\left(\max_{n_1 \leq n_2 \leq n_1+\delta n} |X_{n_2} - X_{n_1}| \geq \varepsilon\sqrt{n} \text{ and } |X_{n_1}| \leq R\sqrt{n}\right) \leq \frac{2\eta}{3}.$$

We shall show that

$$\frac{1}{\delta}\mathbb{P}\left(\max_{n_1 \leq n_2 \leq n_1+\delta n} X_{n_2} \geq X_{n_1} + \varepsilon\sqrt{n} \text{ and } |X_{n_1}| \leq R\sqrt{n}\right) \leq \frac{\eta}{3}, \tag{7}$$

the lower bound on X_{n_2} is similar. Take n_0 such that $\mathbf{P}(A_{n_0}^c) \leq \frac{\varepsilon}{100R}$. Then by the Ergodic Theorem for large n

$$\sum_{x=-2R\sqrt{n}}^{2R\sqrt{n}} I_{A_{n_0}^c}(T^x\omega) \leq \frac{2\varepsilon}{25}\sqrt{n}$$

where I denotes the indicator function. Hence there exists x such that $X_{n_1} \leq x \leq X_{n_1} + \frac{2\varepsilon}{25}\sqrt{n}$ such that $T^x\omega \in A_{n_0}$. Let σ be the first time after n_1 when $X_\sigma = x$. Applying Lemma 5 with $m = 2$ we get

$$\frac{1}{\delta}\mathbb{P}\left(X_{\sigma+k} - X_\sigma > \frac{23\varepsilon}{25}\sqrt{n}\right) \leq \frac{\gamma_2^+ \delta}{\left(\frac{23\varepsilon}{25}\right)^4} < \eta$$

where the last inequality follows from (6). This proves (7) and completes the proof of Lemma 7. $\square$

Let

$$Z(a,b) = \sum_{(x,s):a \leq x \leq b} \delta(x,s)$$

denote the total amount of cookies stored between a and b. We shall denote by τ_x the first time $X_\tau = x$. Let

$$\hat{\tau}(x,M) = \begin{cases} \tau_{x+M} & \text{if } x \geq 0 \\ \tau_{x-M} & \text{if } x < 0 \end{cases}.$$

The next lemma is a quantitative version of the recurrence results of Zerner (2005, 2006).

Lemma 8. *For each N, ε there exists a number M and a set Ω_M such that $\mathbf{P}(\Omega_M) > 1 - \varepsilon$ and for each $x \in \mathbb{Z}$, for each ω such that $T^x\omega \in \Omega_M$, for each $s \in \mathbb{Z}/L\mathbb{Z}$ we have*

$$\mathbb{P}(Y_n \textit{ visits } (x,s) \textit{ at least } N \textit{ times before } \hat{\tau}(x,M)) \geq 1 - \varepsilon. \tag{8}$$

Proof: To fix our ideas consider the case $x \geq 0$. Thus $\hat{\tau}(x, M) = \tau_{x+M}$.

By ellipticity (condition (B)) it is enough to prove the result with (8) replaced by

$$\mathbb{P}(X_n \text{ visits } x \text{ at least } N \text{ times before } \tau_{x+M}) \geq 1 - \varepsilon.$$

Let $\tilde{\tau}_m$ be the first time strictly greater than τ_x when either $|X_{\tilde{\tau}} - x| = m$ or $X_{\tilde{\tau}} = x$. Pick two numbers p, p' such that $\delta L < p' < p < 1$. We claim that if m_1 is large enough then for most environments

$$\mathbb{P}(X_{\tilde{\tau}_{m_1}} = x) > 1 - p. \tag{9}$$

There are two cases to consider: $X_{\tau_x+1} = x + 1$ and $X_{\tau_x+1} = x - 1$ (the case $X_{\tau_x+1} = x$ is trivial). We consider the first case (the second case is easier).

By Optional Stopping Theorem

$$\mathbb{P}(X_{\tilde{\tau}_{m_1}} = x + m_1 | X_{\tau_x+1} = x + 1) = \frac{\mathbb{E}(C_{\tilde{\tau}_{m_1}} - C_{\tau_x}) + 1}{m_1} \leq \frac{Z(x, x + m_1) + 1}{m_1}.$$

So (9) holds if $Z(x, x + m_1) < m_1 p'$ (observe that we need not impose any restrictions in case $X_{\tau_x+1} = x - 1$). Next

$$\mathbb{P}(X_{\tilde{\tau}_{m_2}} = x + m_2 | X_{\tilde{\tau}_{m_1}} = x + m_1) = \frac{\mathbb{E}(C_{\tilde{\tau}_{m_2}} - C_{\tilde{\tau}_{m_1}}) + m_1}{m_2} \leq \frac{Z(x, x + m_2) + m_1}{m_2}.$$

Thus if $\frac{m_1}{m_2} < \frac{p - p'}{2}$ and $Z(x, x + m_2) < p' m_2$ then

$$\mathbb{P}(X_{\tilde{\tau}_{m_2}} = x + m_2 | X_{\tilde{\tau}_{m_1}} = x + m_1) < p.$$

Thus if both $Z(x, x + m_1) < p' m_1$ and $Z(x, x + m_2) < p' m_2$ then

$$\mathbb{P}(X_{\tilde{\tau}_{m_2}} = x + m_2) < p^2.$$

Inductively let m_k be the smallest number such that

$$m_k > \frac{2}{p - p'} m_{k-1}.$$

Then on $\bigcap_{j=1}^{k} \{Z(x, x + m_j) < p' m_j\}$ we have

$$\mathbb{P}(X_{\tilde{\tau}_{m_k}} = x + m_k) < p^k.$$

Thus on this set

$$\mathbb{P}(X \text{ returns to } x \text{ before } \tau_{x+m_k}) \geq 1 - p^k.$$

Since the amount of cookies between x and $x + m_j$ only decreases between the returns the same argument shows that

$$\mathbb{P}(X \text{ returns to } x \text{ at least } N \text{ times before } \tau_{x+m_k}) \geq (1 - p^k)^N.$$

Choose k so that $(1-p^k)^N > 1-\varepsilon$. Let $M = m_k$ and $\Omega_M = \bigcap_{j=1}^{k} \{Z(0, m_j) \leq p' m_j\}$. Then the Ergodic Theorem implies that if m_1 is large enough then $\mathbf{P}(\Omega_M) \geq 1 - \varepsilon$. □

Lemma 9. *For almost all ω, $\frac{C_n - \alpha r_n}{r_n} \to 0$ in probability.*

Proof: Let $\varepsilon > 0$. Take N such that

$$\sum_{j=N+1}^{\infty} \mathbf{E}(\delta(y,j)) < \frac{\varepsilon}{L}.$$

Split $C_n = C_n^- + C_n^+$, where

$$C_n^- = \sum_k \bar{\Delta}_k I(l_k \le N), \quad C_n^+ = \sum_k \bar{\Delta}_k I(l_k > N).$$

By ergodicity we have $C_n^+ \le 2\varepsilon r_n$ for large n so the main contribution comes from C_n^-. Next

$$C_n^- = \sum^{*} \sum_{j=1}^{N} \delta(y,j) I(Q(y,j,n))$$

where $Q(y,j,n)$ is the event that Y visits y at least j times before time n and the meaning of $\sum^{*}$ is the same as in (5). Take a large number M (the precise conditions on M will be given in equations (15) and (17) below) and split $C_n^- = C_n^\partial + C_n^i$ where C_n^∂ contains the terms $y = (x,s)$ where x is within distance M from either maximum or minimum of $X_k, k \le n$ and C_n^i contains the remaining terms. Then $C_n^\partial \le 2LMN$ since there are $2LM$ sites within distance M from either maximum or minimum of $X_k, k \le n$ and for each site only the first N visits give a non-zero contribution to C_n^-. On the other hand

$$C_n^i = \sum^{**} \sum_{j=1}^{N} \delta(y,j) - \sum^{**} \sum_{j=1}^{N} \delta(y,j) I(Q^c(y,j,n)) \tag{10}$$

where the summation in (**) runs over y with

$$\min_{k \le n}(X_k) + M \le x(y) \le \max_{k \le n}(X_k) - M$$

Due to ergodicity for large n

$$\left| \sum^{**} \sum_{j=1}^{N} \delta(y,j) - [L \sum_{j=1}^{N} \mathbf{E}(\delta(y,j))] r_n \right| \le \varepsilon r_n$$

and by the choice of N, $L \sum_{j=1}^{N} \mathbf{E}(\delta(y,j))$ within ε from α. The second term in (10) is less than

$$\hat{C}_n = \sum^{**} \sum_{j=1}^{N} I(\hat{Q}(y,j,M))$$

where $\hat{Q}((x,s),j,M)$ is the event that the j-th visit to (x,s) occurs after time $\hat{\tau}(x,M)$. Therefore to complete the proof of Lemma 9 it remains to show that for almost every ω given ε there exists M such that for large n we have

$$\mathbb{P}(\hat{C}_n > \varepsilon r_n) < \varepsilon. \tag{11}$$

To this end we show that there exists η such that

$$\mathbb{P}(r_n < \eta\sqrt{n}) < \frac{\varepsilon}{3}. \tag{12}$$

Indeed $X_n = B_n + C_n$ and by the Ergodic Theorem for almost every ω there is a constant $K(\omega)$ such that for all n we have

$$0 < C_n < r_n + K(\omega).$$

Since we also have $|X_n| \leq r_n$ the inequality $r_n < \eta\sqrt{n}$ implies that $|B_n| < 2\eta\sqrt{n} + K(\omega)$ but by Lemma 6 $\mathbb{P}(|B_n| < 2\eta\sqrt{n} + K(\omega))$ can be made as small as we wish by taking η small. This proves (12).

Next, by Lemmas 4 and 5

$$\mathbb{P}(r_n > R\sqrt{n}) < \frac{\varepsilon}{3} \tag{13}$$

in R, n are sufficiently large. Combining (12) and (13) we get

$$\mathbb{P}\left(\frac{\hat{C}_n}{r_n} \leq \frac{\sum_{|x(y)|<R\sqrt{n}} \sum_{j=1}^{N} I(\hat{Q}(y,j,M))}{\eta\sqrt{n}}\right) < \frac{2\varepsilon}{3}. \tag{14}$$

Observe that by Lemma 8 we can choose M so large that

$$\mathbb{P}(\hat{Q}((x,s),j,M)) \leq \frac{\varepsilon^2\eta}{100RNL} + I(\Omega_M^c(T^x\omega)). \tag{15}$$

Therefore

$$\mathbb{E}\left(\sum_{|x(y)|<R\sqrt{n}} \sum_{j=1}^{N} I(\hat{Q}(y,j,M))\right) \leq \frac{\varepsilon^2\eta\sqrt{n}}{50} + LN \sum_{|x|<R\sqrt{n}} I(\Omega_M^c(T^x\omega)). \tag{16}$$

By Lemma 8 we can take M so large that

$$\mathbf{P}(\Omega_M^c) \leq \frac{\varepsilon^2\eta}{200RN}. \tag{17}$$

Then by ergodicity the last term in (16) is less than $\frac{\varepsilon^2\eta\sqrt{n}}{50}$ provided that n is sufficiently large. Hence

$$\mathbb{E}\left(\sum_{|x(y)|<R\sqrt{n}} \sum_{j=1}^{N} I(\hat{Q}(y,j,M))\right) \leq \frac{\varepsilon^2\eta\sqrt{n}}{25}.$$

Therefore by Markov inequality

$$\mathbb{P}\left(\sum_{|x(y)|<R\sqrt{n}} \sum_{j=1}^{N} I(\hat{Q}(y,j,M)) > \varepsilon\eta\sqrt{n}\right) < \frac{\varepsilon}{25}.$$

In view of (14) this completes the proof of (11). Lemma 9 follows. □

Proof of Theorem 1: We have

$$\mathcal{W}_n(t) = \mathcal{B}_n(t) + \mathcal{C}_n(t) \tag{18}$$

where $\mathcal{B}_n(t)$ and $\mathcal{C}_n(t)$ are rescaled versions of the martingale and compensator parts of X_n respectively. By Lemma 7 $\{\mathcal{W}_n\}$ is tight, by Lemma 6 $\{\mathcal{B}_n\}$ is tight. Since $\mathcal{C}_n$ is a difference of two tight processes it is tight. Accordingly the triple $\{(\mathcal{W}_n, \mathcal{B}_n, \mathcal{C}_n)\}$ considered as a family of $\mathbb{R}^3$ valued processes is tight. Let $(\mathcal{W}, \bar{\mathcal{B}}, \mathcal{C})$ denote a weak limit of $(\mathcal{W}_n, \mathcal{B}_n, \mathcal{C}_n)$.

By Lemma 6 $\bar{\mathcal{B}}(t) = \mathcal{B}(t)$. By (18) we have

$$\mathcal{W}(t) = \mathcal{B}(t) + \mathcal{C}(t).$$

Therefore it remains to show that

$$\mathcal{C}(t) = \alpha \left[\max_{[0,t]} \mathcal{W}(s) - \min_{[0,t]} \mathcal{W}(s) \right] \tag{19}$$

since this implies that $\mathcal{W}(t)$ satisfies (1) and we will be done by Chaumont and Doney (1999).

Hence given $\varepsilon > 0$ there exists N such that

$$\mathbb{P}\left(\max_{|t_2 - t_1| < 1/N} |\mathcal{C}_n(t_2) - \mathcal{C}_n(t_1)| \geq \varepsilon \right) \leq \varepsilon.$$

Consequently to establish (19) it is enough to show that for each N, ε

$$\mathbb{P}\left(\exists j < N\mathbf{T} \text{ such that } \left| C_n\left(\frac{j}{N}\right) - \alpha \left[\max_{[0,j/N]} \mathcal{W}_n(s) - \min_{[0,j/N]} \mathcal{W}_n(s) \right] \right| > \varepsilon \right) \to 0.$$

Before rescaling this amounts to showing that

$$\mathbb{P}\left(\left| C_{m_j} - \alpha r_{m_j} \right| \leq \varepsilon \sqrt{n} \text{ for } j = 1 \ldots N \right) \to 1$$

where $m_j = nj/N$. Notice that $r_{m_j} \leq r_n$ and by Lemmas 4 and 5 $\mathbb{P}(r_n \geq R\sqrt{n})$ can be made as small as we wish by choosing R and n large. Hence it suffices to check that

$$\mathbb{P}\left(\left| C_{m_j} - \alpha r_{m_j} \right| \leq \varepsilon r_{m_j} \text{ for } j = 1 \ldots N \right) \to 1. \tag{20}$$

However for fixed N, m_j runs over a set of finite cardinality N and so (20) follows from Lemma 9. This concludes the proof of (19). Theorem 1 is established. □

References

D. Aschenbrenner. Master's thesis, University of Tubingen (2010).

A.-L. Basdevant and A. Singh. On the speed of a cookie random walk. *Probab. Theory Related Fields* **141** (3-4), 625–645 (2008a). MR2391167.

A.-L. Basdevant and A. Singh. Rate of growth of a transient cookie random walk. *Electron. J. Probab.* **13**, no. 26, 811–851 (2008b). MR2399297.

P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition (1999). ISBN 0-471-19745-9. A Wiley-Interscience Publication.

L. Chaumont and R. A. Doney. Pathwise uniqueness for perturbed versions of Brownian motion and reflected Brownian motion. *Probab. Theory Related Fields* **113** (4), 519–534 (1999). MR1717529.

P. Hall and C. C. Heyde. *Martingale limit theory and its application*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York (1980). ISBN 0-12-319350-8. Probability and Mathematical Statistics.

E. Kosygina and T. Mountford. Limit laws of transient excited random walks on integers. *Ann. Henri Poincare, Prob. & Stat.* **47**, no. 2, 575–600 (2011). ISSN 0246-0203.

E. Kosygina and M. P. W. Zerner. Positively and negatively excited random walks on integers, with branching processes. *Electron. J. Probab.* **13**, no. 64, 1952–1979 (2008). MR2453552.

T. Mountford, L. P. R. Pimentel and G. Valle. On the speed of the one-dimensional excited random walk in the transient regime. *ALEA Lat. Am. J. Probab. Math. Stat.* **2**, 279–296 (2006). MR2285733.

M. P. W. Zerner. Multi-excited random walks on integers. *Probab. Theory Related Fields* **133** (1), 98–122 (2005). MR2197139.

M. P. W. Zerner. Recurrence and transience of excited random walks on $\mathbb{Z}^d$ and strips. *Electron. Comm. Probab.* **11**, 118–128 (electronic) (2006). MR2231739.