Multi-type TASEP in discrete time

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Abstract. The TASEP (*totally asymmetric simple exclusion process*) is a basic model for a one-dimensional interacting particle system with non-reversible dynamics. Despite the simplicity of the model it shows a very rich and interesting behaviour. In this paper we study some aspects of the TASEP in discrete time and compare the results to the recently obtained results for the TASEP in continuous time. In particular we focus on stationary distributions for multi-type models, speeds of second-class particles, collision probabilities and the "speed process". In discrete time, jump attempts may occur at different sites simultaneously, and the order in which these attempts are processed is important; we consider various natural update rules.

1. Introduction

The TASEP in continuous time was introduced by Spitzer (1970) and can be described as follows. It is a Markov process $(\eta_t)_{t \geq 0}$ on the state space $E = \{0, 1\}^{\mathbb{Z}}$ where for $x \in \mathbb{Z}$ we have that site x is occupied with a particle at time t iff $\eta_t(x) = 1$. Otherwise we say that site x is empty at time t. Starting from some initial configuration $\eta_0 \in E$, *updates* occur at each site as a Poisson process of rate 1, independently; when an update occurs at site x, if there is a particle at site x and a hole to its right at site $x + 1$, the particle jumps from site x to site $x + 1$. If site x is empty, or if site $x + 1$ is already occupied, the update has no effect.

In the model in discrete time, updates occur with some probability $\beta \in (0, 1)$ at each site at each time-step. Since updates occur simultaneously, we now have to choose an order in which to update the sites. We will consider sequential updates (from right to left or from left to right) and sublattice parallel updates (even sites first then odd sites).

For the model in continuous time there exists a vast amount of literature. For an introduction and background to the topic see the books Liggett (1985) (pp. 361-417) and Liggett (1999) (pp. 209-316). However, in some physical models of interest it might be more natural to use a discrete time scale. For example in traffic models we can consider the reaction time of individuals as a smallest time scale (Blythe and Evans, 2007, Chowdhury et al., 2000 and Helbing, 2001) and this suggests modelling traffic with a model in discrete time. The ASEP (*asymmetric simple exclusion process*, particles jump to the right at rate p and to the left at rate $q < p$) in discrete time was studied for example in Schütz (1993), Hinrichsen (1996), Rajewsky et al. (1998) and Blythe and Evans (2007). However, the behaviour of the models in discrete time has not been analysed in as much depth as the model in continuous time. The papers mentioned above are mainly concerned with the model on a finite interval with open boundary conditions and just one type of particles and analyse density profiles and stationary distributions.

In this paper we derive further results for the TASEP in discrete time that correspond to recently obtained results for the continuous-time model. These include stationary distributions for multi-type systems (e.g. Ferrari and Martin, 2006, 2007), laws of large numbers for the path of a second class particle and their connection to competition interfaces in competition growth models (e.g. Ferrari and Pimentel, 2005; Ferrari et al., 2009), and the *TASEP speed process* recently studied by Amir, Angel and Valkó (Amir et al., 2011).

We find that the multi-type invariant distributions for the models with sequential updates are identical with those for the model in continuous time, and do not depend on the parameter β. This has the surprising consequence that various collision probabilities for different particles in a multi-type processes started out of equilibrium, of the sort considered by Ferrari et al. (2009) and Amir et al. (2011), are also independent of β and coincide with the values for a continuous-time process. These probabilities correspond to survival probabilities of clusters in the associated multi-type competition growth models. At the moment, the only argument we have for this property is indirect, using the fact that the set of invariant measures is identical for all β; we do not know of a more direct argument based on local dynamics or couplings.

By contrast, in the case of sublattice-parallel updates, the value of β plays an important role in the set of stationary distributions. All the subsequent results will also heavily depend on the value of the paramter β. We extend the queue-based construction of the multi-type stationary distributions from Ferrari and Martin (2006, 2007) by incorporating queues whose arrival and service rates are different at even and odd times. As Hinrichsen (1996) points out, models with parallel dynamics arrise in traffic and reaction-diffusion models.

The paper is organized as follows. In Section 2 we will give a more formal definition of the model and introduce the multi-type TASEP. The main results are described in Section 3, including results concerning invariant measures and hydrodynamic limits for single-type models which are required in order to state and understand the multi-type models described above. The proofs or proof sketches for the novel results are found in Section 4. In Section 5 we make some brief remarks about a related discrete-time TASEP model with "fully parallel updates".

2. Model

2.1. *Models in continuous and discrete time.* The TASEP in continuous time can be described by its generator L. For cylinder functions $f : E \to \mathbb{R}$ we have

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} \eta(x) \left(1 - \eta(x+1)\right) \left[f\left(\eta^{x,x+1}\right) - f\left(\eta\right) \right]$$

with the configuration $\eta^{x,x+1}$ defined by

$$\eta^{x,x+1}(y) = \begin{cases} \eta(y) & y \notin \{x, x+1\} \\ \eta(x+1) & y = x \\ \eta(x) & y = x+1 \end{cases}$$

Following ideas from Harris (1978) we can use the following graphical construction for the TASEP. Let $\{(P_t^x)_{t \geq 0} : x \in \mathbb{Z}\}$ be a family of independent mean 1 Poisson processes on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For $x \in \mathbb{Z}$ the process P^x marks possible jumps from site x: If $P_t^x - P_{t-}^x = 1$ and $\eta_{t-}(x) = 1$ then the particle at x tries to jump one step to the right at time t. The jump is successful if the adjacent site $x+1$ was unoccupied, i.e. $\eta_{t-}(x+1) = 0$. Note that for every $t > 0$ with positive probability (e^{-t}) there was no jump in the Poisson process P^x up to time t. Since all the Poisson processes are independent there will be infinitely many sites x such that there were no jumps in P^x. These sites separate $\mathbb{Z}$ into intervals of finite length. Since no particle can have crossed the boundaries of these intervals, it is enough to be able to construct the process separately on each of these finite intervals.

We can use the same graphical construction to define the TASEP in discrete time. All we have to do is replace the family of Poisson processes with a family $\{(B_n^x)_{n \geq 0} : x \in \mathbb{Z}\}$ of independent Bernoulli processes with parameter $\beta \in (0, 1)$ and decide on an update rule for the sites. As mentioned in the introduction we will mainly consider the following three update rules:

- Rule R1: Updates are processed in order from right to left.
- Rule R2: Updates are processed in order from left to right.
- Rule R3: All updates at even sites are processed before all updates at odd sites.

To highlight the difference between the three update rules we can look at the following example. Say we are at time n in the configuration displayed in Figure 2.1, with particles at sites -1 and 0 and holes at sites 1 and 2. There are jump attempts at the sites marked with a $*$. The resulting configurations under the three different update rules are as shown in Figures 2.2–2.4.

Note that in R2, a single particle may jump several times at the same time-step (but jumps are only possible onto sites that were already empty at the beginning of the time-step). In R1, several neighbouring particles may jump together at the same time-step. There is a natural symmetry between systems R1 and R2 - one is transformed into the other by exchanging left and right and exchanging the roles of particle and hole. For the last example (R3) the parity of the sites is important.

In connection with the *speed process* we will also mention the model with odd/even updates (R4). Again this can be obtained from R3 by a simple transformation.
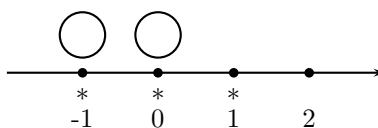
FIGURE 2.1. Configuration at time n and jump marks
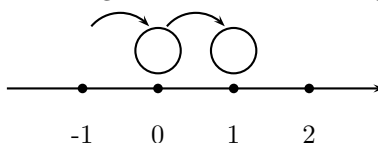
FIGURE 2.2. Configuration at time $n+1$ if we apply R1
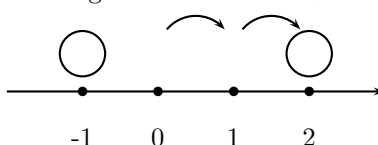
FIGURE 2.3. Configuration at time $n+1$ if we apply R2
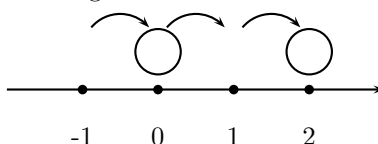
FIGURE 2.4. Configuration at time $n+1$ if we apply R3

As seen above, each of these models shows a slightly different behaviour, but if we rescale time by a factor β^{-1} and let $\beta \to 0$ then they converge to the model in continuous time. In this sense the model in discrete time is more general than the model in continuous time (which in the following we will denote by R0) since we can recover the model in continuous time from the model in discrete time. In discrete time we can also consider the model with (fully) parallel updates where all sites are updated simultaneously. However, many of the methods developed for the model in continuous time that work in the models R1-R3 fail in this case. We will mention some questions connected to this model in Section 5.

2.2. *Percolation representations.* Both in continuous and in discrete time, one important feature of the TASEP is its connection to *last-passage percolation* and the *corner growth model.* Here we consider a special case which corresponds to a particular initial condition of the TASEP, in which, at time 0, all non-positive sites $x \leq 0$ contain a particle and all positive sites $x > 0$ are empty. We label the particles from right to left, so that for $i \geq 1$, particle i starts at site $-i+1$ at time 0 (and always remains to the right of particle $i+1$).

For $n, k \geq 1$, let $T(n, k)$ be the time that particle k jumps to its right for the nth time. Then it is well-known that the variables $T(n, k)$ satisfy the recursions

$$T(n,k) = \max\{T(n-1,k), T(n,k-1)\} + v(n,k) \qquad n, k \geq 1 \qquad (2.1)$$

with boundary conditions $T(0, k) = T(n, 0) = 0$ for all n, k, where $v(n, k)$ are i.i.d. exponential random variables with mean 1. The interpretation is that before particle k can make its nth jump, both particle k must have made its $(n-1)$st

jump, and particle $k - 1$ must have made its nth jump. Once these two events have happened, an amount of time which is exponentially distributed with rate 1 passes before particle k makes its nth jump; this is the random variable $v(n, k)$.

The random variables $T(n, k)$ have an interpretation in terms of last-passage percolation times. For an increasing path π from $z \in \mathbb{Z}_+^2$ to $z' \in \mathbb{Z}_+^2$, i.e. a path with increments in $\{(0, 1), (1, 0)\}$, define the weight of π by

$$S(\pi) = \sum_{z'' \in \pi} v(z'').$$

Write $\Pi(z, z')$ for the set of all increasing paths from z to z'; then

$$R(z, z') = \max_{\pi \in \Pi(z, z')} S(\pi) \tag{2.2}$$

is the weight of the heaviest path from z to z'. Then, via the recursions (2.1), it is easy to see that $T(n, k) = R((1, 1), (n, k))$. In this setting we may interpret the random variable $v(n, k)$ as a weight at the lattice point (n, k).

We turn to the discrete-time case. Now let $w(n, k)$ be i.i.d. random variables whose distribution is geometric with parameter $\beta \in (0, 1)$ (by which we mean that $\mathbb{P}[w(z) = k] = (1 - \beta)^k \beta$ for $k = 0, 1, 2, \ldots$). We define passage-times $\tilde{T}(n, k)$ analogous to $T(n, k)$ above by the recursions

$$\tilde{T}(n, k) = \max\left\{\tilde{T}(n - 1, k), \tilde{T}(n, k - 1)\right\} + w(n, k) \qquad n, k \geq 1.$$

We will describe three variants on these recursions, which pertain to the different update rules R1, R2 and R3. As above, $w(n, k)$ will correspond to the delay before particle k makes its nth jump, once it is free to do so. For $i = 1, 2, 3$, let $T^{(i)}(n, k)$ be the nth jump of particle k under update rule Ri with boundary conditions $T^{(i)}(0, k) = T^{(i)}(n, 0) = -1$ for all n, k.

Rule R1 (updates from right to left)

- Recursions:

$$T^{(1)}(n, k) = \max\left\{T^{(1)}(n - 1, k) + 1, T^{(1)}(n, k - 1)\right\} + w(n, k)$$
$$= \tilde{T}(n, k) + n - 1 \tag{2.3}$$

- In accordance with the updates from right to left, particles k and $k - 1$ can make their nth jumps at the same time-step, but two jumps by the same particle must be separated by at least one time-step.
- This corresponds to a percolation model in which as well as weights $w(n, k)$ at the vertices $(n, k) \in \mathbb{Z}_+^2$, we have weights of size 1 on each horizontal edge between $(n - 1, k)$ and (n, k).

Rule R2 (updates from left to right)

- Recursions:

$$T^{(2)}(n, k) = \max\left\{T^{(2)}(n - 1, k), T^{(2)}(n, k - 1) + 1\right\} + w(n, k)$$
$$= \tilde{T}(n, k) + k - 1 \tag{2.4}$$

- With updates from left to right, a particle may make several jumps at the same time-step, but at least one time-step must separate the nth jump of particles $k - 1$ and k.

- In the corresponding percolation model, the weights of size 1 are now on the vertical edges of the lattice.

Rule R3 (even updates then odd updates)

- Recursions:

$$
T^{(3)}(n,k) =
\begin{cases}
\max\left\{T^{(3)}(n-1,k)+1, T^{(3)}(n,k-1)+1\right\} + w(n,k) \\
\hspace{8cm} n+k \text{ even} \\
\\
\max\left\{T^{(3)}(n-1,k), T^{(3)}(n,k-1)\right\} + w(n,k) \\
\hspace{8cm} n+k \text{ odd}
\end{cases}
$$

$$
=
\begin{cases}
\tilde{T}(n,k) + \frac{n+k-2}{2} & n+k \text{ even} \\
\tilde{T}(n,k) + \frac{n+k-3}{2} & n+k \text{ odd}
\end{cases}
\tag{2.5}
$$

- Now the edge weights of size 1 are added to all edges with an upper/right point (n,k) such that $n+k$ is even.

For the model in continuous time we have, for $x > 0$,

$$
\lim_{n\to\infty} \frac{T([xn],n)}{n} = \left(\sqrt{x}+1\right)^2 \qquad \text{a.s.} \tag{2.6}
$$

This was essentially first shown in Rost (1981). Replacing the exponential weights by geometric weights gives

$$
\lim_{n\to\infty} \frac{\tilde{T}([xn],n)}{n} = \frac{(1-\beta)x + 2\sqrt{(1-\beta)x} + (1-\beta)}{\beta} \qquad \text{a.s.;} \tag{2.7}
$$

see for example O'Connell (2000). Using (2.3)-(2.5), this can easily be used to give similar laws of large numbers for $T^{(i)}([xn],n)$, $i=1,2,3$.

We may also view the system as a growth model. For the continuous-time case,

$$
G_t = \left\{(x,y) \in \mathbb{Z}_+^2 : T(x,y) \le t\right\} \tag{2.8}
$$

be the set of vertices whose passage-time is less than t. This gives a cluster in $\mathbb{Z}_+^2$ which grows over time; there is a 1-1 correspondence relating G_t to the configuration of the TASEP at time t; the length of the row at height $k \in \mathbb{Z}_+$ is the number of jumps particle k has made in the TASEP. In a similar way we can define $G^{(1)}(t)$, $G^{(2)}(t)$ and $G^{(3)}(t)$ by replacing T in (2.8) by $T^{(1)}$, $T^{(2)}$ or $T^{(3)}$ respectively.

2.3. *Multi-type models.* In the *multi-type TASEP* each particle belongs to a class $y \in \mathbb{Z}$ (or more generally $y \in \mathbb{R}$). All particles can still jump into unoccupied sites. When a particle of class k tries to jump into a site which is occupied by a particle of class j two things can happen: If $k \ge j$ the jump is suppressed and if $k < j$ then the two particles swap. This means that the lower the class of a particle the higher is its priority.

An N-type TASEP (containing N classes of particles and holes) can be regarded as a coupling of N ordered single-type TASEPs. If η_0^1, ..., η_0^N are N TASEP configurations such that $\eta_0^1(x) \le \ldots \le \eta_0^N(x)$ for all $x \in \mathbb{Z}$, we can use the same Poisson or Bernoulli processes (this is called *basic coupling*) to get a joint realization of the TASEPs η^1, ..., η^N.

The basic coupling preserves the ordering between the processes (since the updates are processed one by one, this is true for the discrete-time models just as in

the continuous-time case). Thus we can define a multi-type process ξ by

$$\xi_t(x) = N + 1 - \sum_{k=1}^{N} \eta_t^k(x).$$

We write $\xi_t = R\eta_t$. Particles of class k occur at sites x where $\xi(x) = k$. For $k > 1$, these sites represent discrepancies between the processes η^{k-1} and η^k. We may regard particles of type $N + 1$ as holes. Then ξ behaves like a multi-type TASEP with N classes of particles and holes. See for example Ferrari and Martin (2006) for further details.

3. Results

We will divide this section into three subsections: The first deals with invariant measures for single- and multi-type models, the second with hydrodynamic limits and the third with multi-type models out of equilibrium.

3.1. *Invariant measures.*

Proposition 3.1. *For the TASEP in continuous time as well as the discrete time TASEPs R1 and R2, the Bernoulli product measures ν_ρ with marginals $\rho \in [0,1]$ are the only translation invariant stationary ergodic measures with constant marginals. For the TASEP R3, the Bernoulli product measures μ_ρ with marginals $\rho \in [0,1]$ on even sites and marginals $\frac{\rho(1-\beta)}{1-\rho\beta}$ on odd sites are the only stationary ergodic measures with marginals that are translation invariant under even shifts.*

Remark 3.2. Interestingly, the marginals of the invariant Bernoulli product measures for the models R1 and R2 do not depend on the model parameter β, and coincide with the invariant measures for the model in continuous time. In the model R3 however, the densities at even and odd sites differ (with a specific relation between them) and the measure depends on the parameter β.

Proof: For references see for example Liggett (1985) for R0, Blythe and Evans (2007) for R1, R2 and Rajewsky, Santen, Schadschneider and Schreckenberg (Rajewsky et al., 1998) for R3. The uniqueness statements can be proved following the approach of Mountford and Prabhakar (1995). $\qquad\square$

We now turn to the construction of invariant measures for systems with more than one class of particles. We use the construction based on a system of queues in tandem developed in Ferrari and Martin (2006), and begin by recalling notation from that paper.

Given two processes α_1 and α_2, taking values in $\{0,1\}^{\mathbb{Z}}$ and representing the arrival and service processes of a queue respectively, let $D(\alpha_1, \alpha_2)$ be the process of departures from the queue. Now define $D^{(1)}(\alpha) = \alpha$, $D^{(2)}(\alpha_1, \alpha_2) = D(\alpha_1, \alpha_2)$, and recursively $D^{(n)}(\alpha_1, \ldots, \alpha_n) = D\left(D^{(n-1)}(\alpha_1, \ldots, \alpha_{n-1}), \alpha_n\right)$ for $n > 2$. (The process $D^{(n)}$ can be seen as the departure process from a system of $n - 1$ queues in tandem). Now for $\alpha = (\alpha_1, \ldots, \alpha_n)$ we can define a system of n ordered single-type TASEP configurations, denoted $T\alpha = \eta = \left(\eta^1, \ldots, \eta^n\right)$ by $\eta^k = D^{(n-k+1)}(\alpha_k, \ldots, \alpha_n)$. Then the corresponding multi-type configuration $\xi = \xi^{(1,\ldots,n)}$ is given by $\xi = R\eta = RT\alpha$, with $\xi(x) = n + 1 - \sum_{k=1}^{n} \eta^k(x)$ (as in the
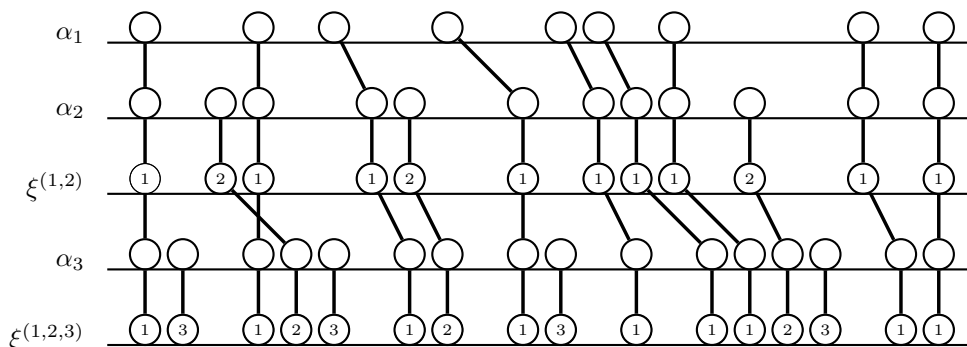
FIGURE 3.5. Queues in tandem and multi-type configurations $\xi^{(1,2)}$
and $\xi^{(1,2,3)}$

last paragraph of Section 2). See Remark 3.4 below for further explanation of the
construction.

We can now state the main result.

To state this result, we work with systems with jumps *from right to left*. To return
to the systems defined before, one simply takes the space-reversal $(\tilde{\eta}_t(x) = \eta_t(-x))$.
Note that time in the queueing system corresponds to space in the particle system.

Theorem 3.3. *If $\alpha = (\alpha_1, \ldots, \alpha_n)$ has distribution $\nu = \nu_{\rho_1} \times \ldots \times \nu_{\rho_n}$ ($\mu = \mu_{\rho_1} \times \ldots \times \mu_{\rho_n}$ respectively for model R3) with $\rho_1 < \ldots < \rho_n$, then the law of $T\alpha = \eta$ is invariant for the coupled multi-line TASEPs R0,R1 and R2 (R3 respectively) and the law of $RT\alpha = R\eta = \xi$ is invariant for the multi-type TASEPs R0, R1 and R2 (R3 respectively) with jumps from right to left. These are the unique stationary translation invariant (invariant under even shifts respectively) ergodic measures with density ρ_1 of first class particles (density ρ_1 of first class particles on even sites), density $\rho_2 - \rho_1$ of second class particles (density $\rho_2 - \rho_1$ of second class particles on even sites), etc.*

Remark 3.4. The mechanism to construct an invariant distribution as described
above can be depicted in the following way: Take α_1 as the arrival process and
α_2 as the service process of a queue. Using α_1 and α_2 we can construct a process
consisting of the departures from this queue (first class particles), unused services
(second class particles) and times when no service was offered (holes). We then use
this process as the arrival process for a queue with service process α_3 where first
class particles have priority over second class particles: If there is a service and a
first and a second class particle are waiting in the queue then the first class particles
gets served first. In this way we get a resulting process consisting of departures of
first class particles (first class particles), departures of second class particles (second
class particles), unused services (third class particles) and holes. Now we can feed
this process into a queue with service process α_4 and so on. If $\alpha = (\alpha_1, \ldots, \alpha_n)$
has distribution $\nu = \nu_{\rho_1} \times \ldots \times \nu_{\rho_n}$ ($\mu = \mu_{\rho_1} \times \ldots \times \mu_{\rho_n}$ respectively) then the
distribution of the resulting multi-type configuration is invariant for the multi-type
TASEP. See Figure 3.5 for an illustration. Note that for models R0, R1, R2, the

queues involved are simply $M/M/1$ queues in discrete-time; the same is almost true for R3, except that we have different arrival and service rates at odd and even times.

Remark 3.5. We observe again that the invariant measures for the multi-type TASEPs R1 and R2 are the same as the invariant measures for the multi-type TASEP in continuous time and that they do not depend on β. Since the invariant measures for the single-type TASEP R3 depend on β the same is true for the invariant measures for the multi-type TASEP R3.

3.2. *Hydrodynamic limits.* We now move to considering systems out of equilibrium. We consider the particular initial configuration given by

$$\eta_0(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x \geq 1 \end{cases}$$

This "step" initial condition corresponds to the corner growth model and to the particular initial conditions for the percolation models described in Section 2.2. We define the following functions f_0, f_1, f_2 and f_3, which will describe the evolving density profile for the continuous-time TASEP and for the TASEPs R1-R3 in discrete time:

$$f_0(u) = \mathbb{1}_{(-\infty,-1]}(u) + \frac{1}{2}(1-u) \cdot \mathbb{1}_{[-1,1]}(u)$$

$$f_1(u) = \mathbb{1}_{(-\infty,-\frac{\beta}{1-\beta}]}(u) + \frac{1}{\beta}\left(1 - \sqrt{\frac{1-\beta}{1-u}}\right) \cdot \mathbb{1}_{[-\frac{\beta}{1-\beta},\beta]}(u)$$

$$f_2(u) = \mathbb{1}_{(-\infty,-\beta]}(u) + \left(1 - \frac{1}{\beta}\left(1 - \sqrt{\frac{1-\beta}{1+u}}\right)\right) \cdot \mathbb{1}_{[-\beta,\frac{\beta}{1-\beta}]}(u) = 1 - f_1(-u)$$

$$f_3(u) = \mathbb{1}_{(-\infty,-\frac{2\beta}{2-\beta}]}(u) + \left(\frac{1}{2} - \frac{u}{\beta}\sqrt{\frac{1-\beta}{4-u^2}}\right) \cdot \mathbb{1}_{[-\frac{2\beta}{2-\beta},\frac{2\beta}{2-\beta}]}(u)$$

We let a_3 be defined by $f_3(u) = \frac{1}{2}\left(a_3(u) + \frac{a_3(u)(1-\beta)}{1-a_3(u)\beta}\right)$, so

$$a_3(u) = \mathbb{1}_{(-\infty,-\frac{2\beta}{2-\beta}]}(u) + \frac{1}{\beta}\left(1 - (2+u)\sqrt{\frac{1-\beta}{4-u^2}}\right) \cdot \mathbb{1}_{[-\frac{2\beta}{2-\beta},\frac{2\beta}{2-\beta}]}(u)$$

For $i = 0, 1, 2, 3$ let $\tau_i(k,t)$ ($t \in \mathbb{R}_+$ or $t \in \mathbb{N}$ respectively) be the distribution of $(\eta_t(k+l), l \in \mathbb{Z})$ in the corresponding model. We have the following result for the TASEP in continuous time and the discrete TASEPs R1-R3.

Theorem 3.6. *For any $u \in \mathbb{R}$ and $i = 0, 1, 2$ the measure $\tau_i([ut],t)$ converges weakly to the Bernoulli product measure with marginals $f_i(u)$ and $\tau_3([ut],t)$ converges weakly to the Bernoulli product measure with marginals $a_3(u)$ on even sites and $\frac{a_3(u)(1-\beta)}{1-a_3(u)\beta}$ on odd sites. In particular we have that for any $u \in \mathbb{R}$ the limit $\lim_{t\to\infty} \mathbb{E}[\eta_t(k)]$ exists and is equal to $f_i(u)$, $i = 0, 1, 2$ depending on which model we are considering, whenever $\frac{k}{t}$ tends to u, and $\lim_{t\to\infty} \mathbb{E}\left[\eta_t(2[\frac{k}{2}])\right] = a_3(u)$ and $\lim_{t\to\infty} \mathbb{E}\left[\eta_t(2[\frac{k}{2}]+1)\right] = \frac{a_3(u)(1-\beta)}{1-a_3(u)\beta}$ in the model R3. Furthermore, for $i = 0, 1, 2, 3$, the quantities $\frac{1}{t}\sum_{ut<k<vt} \eta_t(k)$ converge a.s. to the constant value $\int_u^v f_i(w)dw$, for $u < v$.*
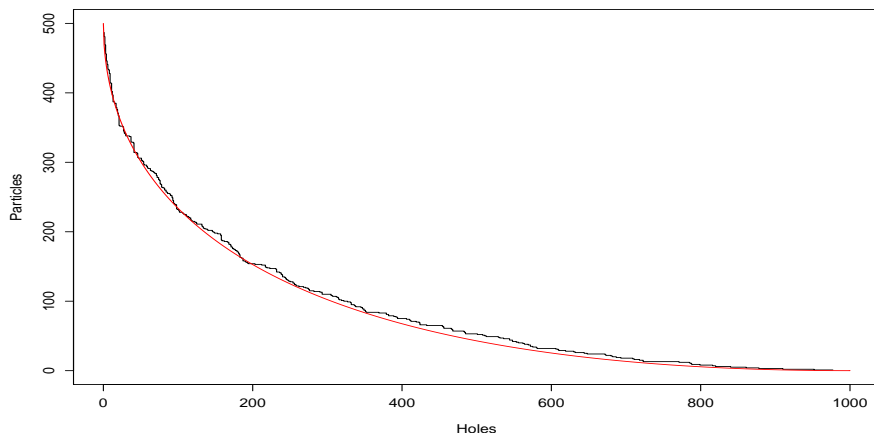
FIGURE 3.6. Black: Simulation of the corner growth model (R2) for
$\beta = 0.5$ and time $n = 1000$; Red: Limiting shape given
by a rescaled version of g_2

The first part of the theorem states convergence to local equilibrium: suitably
rescaled the models converge locally to the unique invariant measures from Theorem
3.1. This implies the other statements of the theorem. However, in the models R0,
R1 and R2 we can prove the second part without proving convergence to local
equilibrium first, while in the model R3 our proof for the second part requires
convergence to local equilibrium. The statements for the model in continuous time
were proved for the first time by Rost (1981). O'Connell (2000) used the connection
between the TASEP and last-passage percolation to prove an equivalent result
about the asymptotic shape of the corner growth model (as defined in Section 2).
The parts of Theorem 3.6 concerning the models in discrete time can be proved
using exactly the same methods as Rost (1981) and O'Connell (2000). In Section
4 we will outline the proof for the model R3.

Remark 3.7. From the convergence to the density profiles f_0, f_1, f_2 and f_3 we can
easily deduce a shape theorem for the corner growth model defined in Section 2.
The asymptotic shape in the models R1, R2 and R3 (after rescaling by t) are for
example given by the functions

$$g_1(x) = \frac{1}{1 - \beta} \left(\sqrt{\beta} - \sqrt{x} \right)^2 \tag{3.1}$$

for $x \in [0, \beta]$,

$$g_2(x) = \left(\sqrt{\beta} - \sqrt{(1 - \beta)x} \right)^2 \tag{3.2}$$

for $x \in [0, \frac{\beta}{1-\beta}]$, see Figure 3.6 (simulation with $\beta = 0.5$ up to time $n = 1000$), and

$$g_3(x) = \frac{1}{(2 - \beta)^2} \left(\sqrt{4x(1 - \beta)} - \sqrt{4\beta - \beta^2 x - 2\beta^2} \right)^2 \tag{3.3}$$

for $x \in \left[0, \frac{2\beta}{2-\beta} \right]$.

Remark 3.8. We may rescale time as well as space in Theorem 3.6, and look at the limit $f(u,t) = \lim_{N\to\infty} \mathbb{E}\left[\eta_{Nt}(k)\right]$ for $\lim_{N\to\infty}\frac{k}{N} = u$. For the continuous-time model, this density profile is governed by Burgers' equation,

$$\frac{\partial f}{\partial t} + \frac{\partial(f(1-f))}{\partial u} = 0; \tag{3.4}$$

the solution, with initial condition $f(u,0) = \mathbb{1}_{(-\infty,0]}(u)$, is

$$f(u,t) = \mathbb{1}_{(-\infty,-t]}(u) + \frac{t-u}{2t} \cdot \mathbb{1}_{[-t,t]}(u)$$

The differential equation also governs the evolution of the density profile for more general initial configurations than the "step" initial condition. We can get equations analogous to (3.4) for the models in discrete time. For example, for model R1,

$$f_1(u,t) = \mathbb{1}_{\left(-\infty,-\frac{\beta t}{1-\beta}\right]}(u) + \frac{1}{\beta}\left(1 - \sqrt{\frac{t(1-\beta)}{t-u}}\right) \cdot \mathbb{1}_{\left[-\frac{\beta t}{1-\beta},\beta t\right]}(u)$$

solves

$$\begin{cases} \frac{\partial f_1}{\partial t} + \frac{\partial}{\partial u}\frac{\beta f_1(1-f_1)}{1-\beta f_1} = 0 \\ f_1(u,0) = \mathbb{1}_{(-\infty,0]}(u) \end{cases} \tag{3.5}$$

Here $\frac{\beta f_1(1-f_1)}{1-\beta f_1} = \sum_{n=1}^{\infty}\beta^n f_1^n(1-f_1)$ is the probability that a particle jumps from a given site to its neighbour in a model in equilibrium with marginal density f_1.

3.3. *Multi-type models out of equilibrium.* In this section we consider multi-type TASEPs $\xi_t \in \mathbb{Z}^{\mathbb{Z}}$ similar to Section 2.3. With the results from Theorem 3.6 we can calculate the distribution of the asymptotic speed of a single second class particle in the TASEP with initial configuration

$$\xi_0(x) = \begin{cases} 1 & x \le -1 \\ 2 & x = 0 \\ 3 & x \ge 1 \end{cases}$$

As the particles of class 3 are weaker than all other particles in the model we can think of these particles as holes. So the second-class particle sees only particles to its left and only holes to its right. The second-class particle can be seen as a discrepancy between two copies of the "step" initial condition considered in the last section, one of which is shifted by one step to the right. Hence the path of the second-class particle corresponds to the propagation of the discrepancy under the basic coupling. The results for the models in discrete time correspond to the result for the model in continuous time first obtained by Ferrari and Kipnis (1995). They prove convergence in distribution. In order to prove a.s. convergence we can use the connection to last-passage percolation and the growth model: As in Ferrari and Pimentel (2005) the path of the second class particle corresponds to a competition interface in the growth model which has a.s. an asymptotic direction.

Theorem 3.9. *For $i = 0,1,2,3$ let $X^{(i)}(t)$ denote the position of the second class particle at time t in the corresponding model. Then we have*

$$\frac{X^{(i)}(t)}{t} \xrightarrow[t\to\infty]{a.s.} U^{(i)}$$

for random variables $U^{(i)}$ with distribution functions $1 - f_i$ for $i = 0,1,2$ and a_3 for $i = 3$.

Remark 3.10. The proofs for convergence in distribution are analogous to those for the model in continuous time (see for example Ferrari and Kipnis, 1995) apart from a small complication in the model R3 with the *particle-particle coupling*. We will give an account of this proof in Section 4. For the almost sure convergence we will explain the construction of the competition interface for the models in discrete time and prove that the second class particle has almost sure an asymptotic speed by using results about semi-infinite geodesics in the percolation models similar to Ferrari and Pimentel (2005). An interesting observation will be that the distribution of the asymptotic direction of the competition interface corresponding to the path of the second class particle is the same in the models R1 and R3, see Remark 4.12. However, this does not imply that the distribution of the speed of the second class particle is the same in the two models.

Remark 3.11. In the continuous model the distribution of the asymptotic speed of the second class particle turns out to be uniform on $[-1, 1]$. The distributions in the models in discrete time are more complicated.

Using Theorem 3.9 we can define the following so-called speed process: Consider the multi-type TASEP with initial configuration $\xi_0(n) = n$. By Theorem 3.9 we know that each particle has a.s. an asymptotic speed U_n: Particle n has only stronger particles to its left and only weaker particles (that can be seen as holes) to its right just like the second class particle in the initial configuration of Theorem 3.9 and therefore we can apply Theorem 3.9 to the speed of every particle. We call the process $\{U_n\}_{n \in \mathbb{Z}}$ the speed process and denote its distribution by μ. This process is stationary, and its marginals (for the various models) are given by the distributions in Theorem 3.9. Furthermore, we write $Y_n(m)$ instead of $\xi_m(n)$ and denote the position of particle n at time m by $X_n(m)$ in order to be consistent with the notation introduced in Amir et al. (2011). They have studied the speed process for the model in continuous time. Note that both Y_n and X_n can be seen as permutations of the set $\mathbb{Z}$, and are inverse to each other. We define $g_i = 1 - f_i$ for $i = 0, 1, 2$ and $g_3 = 1 - a_3$, $g_4 = 1 - \frac{a_3(1-\beta)}{1-a_3\beta}$ and our first result is the following theorem corresponding to Theorem 1.5 of Amir et al. (2011) (note that the labels of particles can now be in $\mathbb{R}$ instead of just $\mathbb{Z}$):

Theorem 3.12. *For $i = 0, 1, 2$, $\mu^{(i)}$, the distribution of the speed process in model Ri, is the unique stationary ergodic measure for the TASEP R0 whose marginals have distribution function g_i. Correspondingly, for $i = 3, 4$, $\mu^{(i)}$ is the unique stationary measure for the TASEP Rj(i), which has marginals distributed according to g_i on even sites and $g_{j(i)}$ on odd sites, where $j(3) = 4$ and $j(4) = 3$.*

For $i = 0$ this gives the result from Amir et al. (2011) saying that the distribution of the speed process is itself a stationary ergodic measure for the TASEP in continuous time (the marginals are uniform on $[-1, 1]$ in this case). The other parts of Theorem 3.12 follow from nice dualities between the models R1 and R2 and between the models R3 and R4, and the fact that R0, R1 and R2 all have the same set of stationary distributions, whatever the value of β (as given in Theorem 3.3). The dualities are given by the following result:

Theorem 3.13. *Consider the starting configuration $Y_n(0) = n$. For $i = 0, 1, 2, 3, 4$ and any fixed $m > 0$ the process $\{X_n^{(i)}(m)\}_{n \in \mathbb{Z}}$ has the same distribution as the process $\{Y_n^{(j(i))}(m)\}_{n \in \mathbb{Z}}$ where $j(0) = 0$, $j(1) = 2$, $j(2) = 1$, $j(3) = 4$ and $j(4) = 3$.*

The following theorems provide some explicit results about the joint distributions of the speeds of adjacent particles (and particles 0 and 2 in model R3). The first result is Theorem 1.7 of Amir et al. (2011). The remaining theorems and remarks give analogous results for the TASEPs R1, R2 and R3.

Theorem 3.14 (TASEP R0). *The joint distribution of (U_0, U_1), supported on $[-1,1]^2$, is*

$$s(x,y)dxdy + r(x)\mathbb{1}_{\{x=y\}}dx$$

with

$$s(x,y) = \begin{cases} \frac{1}{4} & x > y \\ \frac{y-x}{4} & x \le y \end{cases} \quad and \quad r(x) = \frac{1-x^2}{8}$$

In particular, $\mathbb{P}[U_0 > U_1] = \frac{1}{2}$, $\mathbb{P}[U_0 = U_1] = \frac{1}{6}$ and $\mathbb{P}[U_0 < U_1] = \frac{1}{3}$.

Theorem 3.15 (TASEP R1). *The joint distribution of (U_0, U_1) has support on $\left[-\frac{\beta}{1-\beta}, \beta\right]^2$ and is given by*

$$s_1(x,y)dxdy + r_1(x)\mathbb{1}_{\{x=y\}}dx$$

with

$$s_1(x,y) = \begin{cases} \frac{1-\beta}{4\beta^2}(1-x)^{-\frac{3}{2}}(1-y)^{-\frac{3}{2}} = g_1'(x)g_1'(y) & x > y \\ \frac{1-\beta}{2\beta^3}(1-x)^{-\frac{3}{2}}(1-y)^{-\frac{3}{2}}\left(\sqrt{\frac{1-\beta}{1-y}} - \sqrt{\frac{1-\beta}{1-x}}\right) & x \le y \end{cases}$$

and

$$r_1(x) = \left(\frac{\sqrt{1-\beta}}{2\beta^2(1-u)^{\frac{3}{2}}}\left(1 - \frac{1}{\beta}\right) + \frac{1-\beta}{2\beta^2(1-u)^2}\left(\frac{2}{\beta} - 1\right) - \frac{\sqrt{1-\beta}(1-\beta)}{2\beta^3(1-u)^{\frac{5}{2}}}\right)$$

In particular, $\mathbb{P}[U_0 > U_1] = \frac{1}{2}$, $\mathbb{P}[U_0 = U_1] = \frac{1}{6}$ and $\mathbb{P}[U_0 < U_1] = \frac{1}{3}$.

Remark 3.16. By symmetry, the joint distribution of (U_0, U_1) in the model R2 is the same as that of $(-U_1, -U_0)$ in the model R1.

Theorem 3.17 (TASEP R3). *The joint distribution of (U_0, U_1) has support on $\left[-\frac{2\beta}{2-\beta}, \frac{2\beta}{2-\beta}\right]^2$ and is given by*

$$s_2(x,y)dxdy + r_2(x)\mathbb{1}_{\{x=y\}}dx$$

with

$$s_2(x,y) = \begin{cases} g_3'(x)g_4'(y) & x > y \\ g_3'(x)g_4'(y)(g_4(y) - g_4(x)) & \\ \qquad \cdot (2 - g_4(x)\beta - g_4(y)\beta)\left(\frac{2+y}{2-y}\right) & x \le y \end{cases}$$

and

$$r_2(x) = \frac{1-\beta}{\beta^3}\left(\frac{2(2-\beta)}{4-x^2} - \frac{8}{4-x^2}\sqrt{\frac{1-\beta}{4-x^2}}\right)$$

In particular,

$$\mathbb{P}[U_0 > U_1] = \frac{1}{\beta^2}\left(\beta - (1-\beta)\log\left(\frac{1}{1-\beta}\right)\right)$$

$$\mathbb{P}[U_0 = U_1] = \frac{(1-\beta)(2-\beta)}{\beta^3}\log\left(\frac{1}{1-\beta}\right) - \frac{2(1-\beta)}{\beta^2}$$

and

$$\mathbb{P}[U_0 < U_1] = \frac{(1-\beta)(2-\beta)}{\beta^2} + \frac{2(1-\beta)^2}{\beta^3}\log(1-\beta)$$

Remark 3.18. Again by symmetry, we have that under rule R3, (U_1, U_2) has the same distribution as $(-U_1, -U_0)$.

Theorem 3.19 (TASEP R3). *The joint distribution of (U_0, U_2) has support on $\left[-\frac{2\beta}{2-\beta}, \frac{2\beta}{2-\beta}\right]^2$ and is given by*

$$s_3(x, y)dxdy + r_3(x)\mathbb{1}_{\{x=y\}}dx$$

with

$$s_3(x, y) = \begin{cases} g_3'(x)g_3'(y) & x > y \\ g_3'(x)g_3'(y)\left(g_4(x)^2 - g_4(y)^2\right. \\ \qquad -\frac{2(g_4(x)-1)(g_4(y)^2 - 2g_4(x)g_4(y) + g_4(x) - 1)}{1 - \beta g_4(x)} \\ \qquad \left. +\frac{2(g_4(y)-1)(g_4(x)^2 - 2g_4(x)g_4(y) + g_4(y) - 1)}{1 - \beta g_4(y)} - 1\right) & x \le y \end{cases}$$

and

$$r_3(x) = \frac{g_3(u)(1 - g_3(u))(1 - g_4(u)(1 - g_4(u)))}{\frac{(2-u)\beta}{2}\sqrt{\frac{4-u^2}{1-\beta}}}$$

In particular,

$$\mathbb{P}\left[U_0 > U_2\right] = \frac{1}{2}$$

$$\mathbb{P}\left[U_0 = U_2\right] = \frac{1}{6} + \frac{1}{3\beta} - \frac{13}{3\beta^2} + \frac{8}{\beta^3} - \frac{4}{\beta^4} - \frac{(1-\beta)^2}{\beta^3}\left(\log\left(\frac{1}{1-\beta}\right)\right)\left(\frac{2}{\beta} - \frac{4}{\beta^2}\right)$$

and

$$\mathbb{P}\left[U_0 < U_2\right] = \frac{1}{3} - \frac{1}{3\beta} + \frac{13}{3\beta^2} - \frac{8}{\beta^3} + \frac{4}{\beta^4} + \frac{(1-\beta)^2}{\beta^3}\left(\log\left(\frac{1}{1-\beta}\right)\right)\left(\frac{2}{\beta} - \frac{4}{\beta^2}\right)$$

We see that in every model we have that the speeds are independent on the set where $U_0 > U_1$ ($U_0 > U_2$ respectively). This agrees with the result in continuous time. The striking result, shown by Amir et al. (2011) for the continuous model, that with positive probability the two continuous random variables U_0 and U_1 are equal, holds also in the discrete models.

Interestingly, the probabilities $\mathbb{P}\left[U_0 > U_1\right]$, $\mathbb{P}\left[U_0 = U_1\right]$ and $\mathbb{P}\left[U_0 < U_1\right]$ are the same for models R0, R1 and R2, and do not depend on the parameter β. This is rather surprising since β is not just a scaling parameter (i.e. we cannot produce models with different values of β by just applying a time change). In fact, much more is true. From the first part of Theorem 3.3, we see that, although the marginal distribution of each U_i depends on the model and the value of β, we can obtain the distribution for either of R1 and R2 and any value of β by applying an appropriate monotone function to each entry U_i (see the proof of Theorem 3.15 for further details). Hence the relative ordering of the variables U_i is not affected by the model or the value of β.

To go further, consider particles i and j with $i < j$. It's clear that if $U_i < U_j$ then particle i can never overtake particle j, while if $U_i > U_j$ then particle i must overtake particle j. In Amir et al. (2011), it's shown that for the continuous-time model, with probability 1, if $U_i = U_j$ then particle i overtakes particle j. The same result can be shown for the discrete-time models, although the calculations involved in the argument are rather more complicated than those used to prove Theorem 1.14 of Amir et al. (2011), and we omit them here. So, for example, the probability that particle i overtakes particle j is the same for models R0, R1 and

R2. Indeed, more completely one can define an ordering $\prec$ on $\mathbb{Z}$ by $i \prec j$ iff particle j is eventually to the right of particle i. Then we have the following result:

Corollary 3.20. *The ordering $\prec$ has the same distribution for R0, R1 and R2 and for any value of β.*

It would certainly be interesting to have a more direct understanding of this property, based for example on couplings or local dynamics, as well as the indirect argument based on the equivalence of multi-type equilibrium distributions.

Overtaking probabilities in the multi-type TASEP can also be interpreted in terms of questions of survival or extinction in multi-type growth models. In Ferrari et al. (2009) a coupling is given between the multi-type TASEP and a three-type version of the corner growth model, under which a given cluster survives for ever if and only if particle 0 never overtakes particle 1. (The extinction of the cluster occurs if two interfaces in the growth model meet – the paths of these interfaces are related to the paths of the two particles). Different overtaking events in the TASEP can be represented by varying the initial condition in the competition growth model. Using the results above, we find that the survival probabilities in the growth model will remain unchanged if we move from the continuous-time model to natural discrete-time models which correspond to models R1 or R2 in the TASEP. Again, this is certainly not obvious from the local dynamics of the processes.

Unlike in models R1 and R2, in the model R3 the probabilities $\mathbb{P}[U_0 > U_1]$, $\mathbb{P}[U_0 = U_1]$ and $\mathbb{P}[U_0 < U_1]$ do depend on β and the behaviour of the model is qualitatively different for different values of β (Theorem 3.17): For small β we have $\mathbb{P}[U_1 < U_0] > \mathbb{P}[U_1 > U_0]$, but $\mathbb{P}[U_1 < U_0] < \mathbb{P}[U_1 > U_0]$ for large β (the transition occurs at $\beta = 0.38860064568\ldots$).

Note however that for $\beta \to 1$ the probabilities relating U_0 and U_2 in Theorem 3.19 converge to $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$, i.e. to the probabilities we get in the continuous model and R1 and R2 for the speeds of particle 0 and 1. In a sense, for large β the particles 0 and 2 in the model R3 behave like adjacent particles in the models R1 and R2. This can heuristically be seen in the following way: We consider the particles in the model R3 (with large β close to 1) starting on even sites. In general, particles starting on an even site will move two steps to the right in each time-step since β is large and we update even sites first. If a particle does not jump either during the even or the odd update (which happens with probability $2\beta(1 - \beta)$) it ends up on an odd site and starts moving left until either

- (A) it hits a weaker particle to the left by which it cannot be overtaken
- (B) it does not get jumped over either during an even or odd update because the adjacent particle to the left did not try to jump

In both cases the particle itself will return to an even site (with high probability) and resume moving to the right. The particle that caused the stop (either because it was weaker or because it did not try to jump) will itself start moving to the left until (A) or (B) happens. Now consider the model R1 with large β. Most particles will move one step to the right in each time-step, but some particles do not jump and therefore get overtaken until again either (A) or (B) happens (where we remove the part "either during and even or odd update"). Particles in these two models have different speeds, but the probabilities $\mathbb{P}[U_0 > U_1]$, $\mathbb{P}[U_0 = U_1]$, $\mathbb{P}[U_0 < U_1]$ in R1 and $\mathbb{P}[U_0 > U_2]$, $\mathbb{P}[U_0 = U_2]$ and $\mathbb{P}[U_0 < U_2]$ in R3 are (almost) the same.

4. Proofs

4.1. *Invariant Measures.*

Proof of Theorem 3.3: The idea of the proof for Theorem 3.3 for the discrete-time models is the same as for the continuous-time model R0 (see Ferrari and Martin, 2006), but here we will use a slightly different definition for the so-called dual points: In Ferrari and Martin (2006) the authors used an explicit construction that transformed a set ω of Bernoulli points into a corresponding set $\triangle(\omega)$ of dual points. Here we provide a more general method to obtain these dual points. We will not provide an explcit set of dual points, but only explain how to obtain one that has the required properties. When thinking about the model R3 bear in mind that we have different densities on even and odd sites. We start with the following Proposition, that asserts that for every particle density ρ and every set of Bernoulli marks ω there is essentially only one stationary and space-ergodic TASEP trajectory. This corresponds to Proposition 8 in Ferrari and Martin (2006).

Proposition 4.1. *For any particle density $\rho \in (0,1)$ and Bernoulli marks ω we can get an essentially unique stationary and space-ergodic trajectory $(\eta_n, n \in \mathbb{Z})$ for the TASEP governed by ω with time-marginals ν^ρ.*

Remark 4.2. Essentially unique means that if we have two stationary and space-ergodic trajectories then they are the same with probability 1.

Proof: The Proposition can be proved using an argument similar to the one used in Mountford and Prabhakar (1995); see also Ferrari and Martin (2006).   □

Now we define for each of the three models a set of dual points $\triangle_\rho(\omega)$ such that the time reflections of these dual points govern the time reversal of the TASEP. Given the Bernoulli process ω and the trajectory $(\eta_n, n \in \mathbb{Z})$ we define $C_1 \subset \omega$ as the set of jump marks that actually lead to a particle jumping one step to the left (i.e. the site that was updated was occupied and the particle was not blocked). Now we look at the time reversal $(\widetilde{\eta}_n, n \in \mathbb{Z}) = (\eta_{-n}, n \in \mathbb{Z})$ of $(\eta_n, n \in \mathbb{Z})$. Because we want the time reflections of the dual points to govern the time reversal of the TASEP we have to have $C_1 \subset \triangle_\rho(\omega)$.

The time reversal has the same distribution as the space reversal. The space reversal is governed by a Bernoulli process (the space reversal of ω) and particles jump to the right. In particular, all possible jumps happen at rate β. That means that in the time reversal all possible jumps happen at rate β, as well.

Let C_2 be the set of points $x \in \left(\left(\mathbb{Z} + \frac{1}{2}\right) \times \mathbb{Z}\right) \backslash C_1$ for which adding x to $\triangle_\rho(\omega)$ would not change the trajectory of the time reversal (i.e. if we let the time reversal be governed by the time reflections of the dual points then the trajectory does not change if we add or remove x from $\triangle_\rho(\omega)$). Let $\widetilde{\omega}$ be a Bernoulli process with rate β on C_2, independent of everything else. That means $\widetilde{\omega} \subset C_2$ and each point from C_2 is present in $\widetilde{\omega}$ independently with probability β. Now put $\triangle_\rho(\omega) = C_1 \cup \widetilde{\omega}$. The properties of these dual points are stated in the following Proposition that corresponds to Proposition 10 in Ferrari and Martin (2006).

Proposition 4.3. *The time reflections of the dual points $\triangle_\rho(\omega)$ defined above govern the time reversal of $(\eta_n, n \in \mathbb{Z})$ and they form a Bernoulli process with rate β on $\left(\mathbb{Z} + \frac{1}{2}\right) \times \mathbb{Z}$. Furthermore, the set of dual points before time m is independent of the configuration η_m.*

Proof: The first two statements are true by the construction of the dual points. The dual points $\{(x,n) \in \triangle_\rho(\omega) : n < m\}$ govern the evolution of the time reversal on the time interval $(-m, \infty)$ starting from the configuration $\widetilde{\eta}_{-m}$ and they are independent of this configuration. But $\widetilde{\eta}_{-m} = \eta_m$. So the set of dual points $\{(x,n) \in \triangle_\rho(\omega) : n < m\}$ is independent of η_m.   □

We now take $\rho_1 < \ldots < \rho_n$ and let $\alpha_m = (\alpha_m^1, \ldots, \alpha_m^n)$ be the multiline TASEP trajectory governed by ω. This means that $\omega^n = \omega$, $\omega^k = \triangle_{\rho_{k+1}}(\omega^{k+1})$ and $(\alpha_m^k)_{m \in \mathbb{Z}}$ is the TASEP trajectory governed by ω^k with density ρ_k. Then by the independence of the dual points before time m from the configuration η_m we get that the multiline process is stationary with product measure $\nu = \nu_{\rho_1} \times \ldots \times \nu_{\rho_n}$ (Proposition 11 in Ferrari and Martin, 2006). As in the paragraph preceding Theorem 3.3 we define $\eta = (\eta^1, \ldots, \eta^n)$ by $\eta^k = D^{(n-k+1)}(\alpha_k, \ldots, \alpha_n)$. Then induction arguments and some case-by-case checking for $n = 2$ as done in Ferrari and Martin (2007) show that $(\eta_n^k)_{n \in \mathbb{Z}}$ is the TASEP trajectory governed by ω with particle density ρ_k (Proposition 12 in Ferrari and Martin, 2006) and this implies Theorem 3.3.   □

Remark 4.4. As mentioned in the beginning of this section, for the model R3 we have to think of the ρ_k as densities on even sites and we have to replace the ν_{ρ_k} by μ_{ρ_k}.

Remark 4.5. Inherent in the tandem queue construction for the multi-type stationary distribution in model R3 is a version of Burke's theorem for the queues with different arrival and service rates on even and odd sites. Consider a queue with arrival process A_n, service process S_n and departure process D_n. Let A_n be a Bernoulli process with rate $\rho_1 = (\gamma_1, \gamma_2) \in (0,1)^2$ which means that on even sites arrivals happen with probability γ_1 and on odd sites they happen with probability γ_2. Motivated by the invariant distributions for the TASEP R3 with just one type of particles we want γ_1 and γ_2 to satisfy

$$\gamma_2 = \frac{\gamma_1 (1 - \beta)}{1 - \gamma_1 \beta} \tag{4.1}$$

where $\beta \in (0,1)$ is the rate at which jumps in the TASEP happen. Analogously, we let S_n be a Bernoulli process with rate $\rho_2 = (\delta_1, \delta_2) \in (0,1)^2$ where

$$\delta_2 = \frac{\delta_1 (1 - \beta)}{1 - \delta_1 \beta} \tag{4.2}$$

(and $\gamma_1 < \delta_1$, $\gamma_2 < \delta_2$). The main observation in Burke's Theorem (see for example Burke, 1956) that shows that arrival and departure process have the same distribution is that the queue length process is reversible and that departures look in the reversed process like arrivals in the original process. Interestingly, it turns out that in the queueing model described above there exists a stationary reversible distribution π for the queue length process which is independent of whether we just observed arrivals and services at even sites or at odd sites. π is given by

$$\pi(j) = \left(1 - \frac{\gamma_1 (1 - \delta_1)}{(1 - \gamma_1) \delta_1}\right) \left(\frac{\gamma_1 (1 - \delta_1)}{(1 - \gamma_1) \delta_1}\right)^j \qquad j = 0, 1, \ldots$$

and it is reversible because it satisfies the two systems of detailed balance equations

$$\pi(j) \gamma_i (1 - \delta_i) = \pi(j+1)(1 - \gamma_i) \delta_i \qquad j = 0, 1, \ldots$$

for $i = 1, 2$ ($i = 1$ corresponds to even sites, $i = 2$ corresponds to odd sites). This follows from the relations (4.1) and (4.2). As in Burke's Theorem it follows from the reversibility of the queue length process that the departure process has the same distribution as the arrival process, i.e. D_n is a Bernoulli process with rate $\rho_1 = (\gamma_1, \gamma_2)$.

Indeed, the multi-type construction yields extensions of this result which give input-output theorems for the priority queues with more than one type of customer. For a discussion of the analogous result in the context of constant arrival and service rates, see for example Section 6 of Ferrari and Martin (2006).

4.2. Hydrodynamic limits.

Proof outline for Theorem 3.6 for the TASEP R3: The following Propositions correspond to Propositions 2,3 and 5 in Rost (1981) and the proofs are essentially the same as in Rost (1981).

Proposition 4.6. *For every $u \in \mathbb{R}$ the random variables $\frac{1}{n} S([un], n) = \frac{1}{n} \sum_{k > [un]} \eta_n(k)$ converge a.s. and in L^1 to a constant $h_3(u)$, as n goes to infinity. The function h_3 is decreasing, convex; one has $h_3(u) = 0$ for $u > \frac{2\beta}{2-\beta}$ and $h_3(u) = -u$ for $u < -\frac{2\beta}{2-\beta}$.*

Proposition 4.7. *If h_3 is differentiable at u, one has*

$$\lim_{n \to \infty} \mathbb{E}\left[\eta_n \left(2 \left[\frac{k}{2}\right] \right) + \eta_n \left(2 \left[\frac{k}{2}\right] + 1 \right) \right] = -2h_3'(u)$$

whenever $\frac{k}{n}$ tends to u.

Proposition 4.8. *Let $\mu_3(k, n)$ be the distribution of $(\eta_n(k + l), l \in \mathbb{Z})$. If $h_3'(u)$ exists, any weak limit μ_3^* of the measures $\mu_3 \left(2 \left[\frac{un}{2}\right], n \right)$ for $n \to \infty$ is of the form*

$$\mu_3^* = \int_0^1 \tau_x \sigma(dx)$$

with some probability σ on $[0, 1]$. τ_x is the Bernoulli product measure with density $b(x)$ on even sites and density $\frac{b(x)(1-\beta)}{1-b(x)\beta}$ on odd sites where $b(x)$ is such that the average density is given by $x = \frac{1}{2}\left(b(x) + \frac{b(x)(1-\beta)}{1-b(x)\beta} \right)$. That means that from Proposition 4.7 it follows that the measure σ satisfies

$$\int_0^1 x\sigma(dx) = f_3(u) = -h_3'(u)$$

We can use the results from O'Connell (2000) about last-passage percolation (see (2.7)) to calculate the function h_3:

$$h_3(u) = \begin{cases} -u & u \leq -\frac{2\beta}{2-\beta} \\ \frac{1}{\beta}\left(2 - \frac{u\beta}{2} - \sqrt{(4 - u^2)(1 - \beta)} \right) - 1 & -\frac{2\beta}{2-\beta} \leq u \leq \frac{2\beta}{2-\beta} \\ 0 & u \geq \frac{2\beta}{2-\beta} \end{cases}$$

Since h_3 is differentiable we can identify $f_3 = -h'_3$ from Proposition 4.8 as

$$f_3(u) = -h'_3(u) = \begin{cases} 1 & u \leq -\frac{2\beta}{2-\beta} \\ \frac{1}{2} - \frac{u}{\beta}\sqrt{\frac{1-\beta}{4-u^2}} & -\frac{2\beta}{2-\beta} \leq u \leq \frac{2\beta}{2-\beta} \\ 0 & u \geq \frac{2\beta}{2-\beta} \end{cases}$$

As mentioned after Theorem 3.6 in the models R0, R1 and R2 this is enough to prove the convergence statements at the end of Theorem 3.6; this is done using the monotonicity of the distribution of $\eta_n(k)$ in k. However, due to the different behaviour at odd and even sites, this monotonicity does not hold for the model R3, and the average density f_3 does not pick up the density fluctuations between even and odd sites. Hence without knowing that the model converges to local equilibrium (which would allow us to calculate the function a_3 from f_3) we cannot prove the last part of Theorem 3.6. The essential step for proving convergence to local equilibrium is the following proposition, the proof of which is again the same as in Rost (1981) (Proposition 6). Let $\rho(k; F; n) = \mathbb{P}\left[\eta_n(k+i) = 1, i \in F\right]$ for a set $F \subset \mathbb{Z}$.

Proposition 4.9. *For any finite set F and any $\epsilon > 0$ there exists a $\delta > 0$, n_0 such that*

$$\left| \rho\left(2\left[\frac{un}{2}\right]; F; n\right) - \rho\left(2\left[\frac{\overline{u}n}{2}\right]; F; n\right) \right| \leq \epsilon$$

for $|u - \overline{u}| \leq \delta$ and $n \geq n_0$. Also

$$\left| \rho\left(2\left[\frac{\overline{u}n}{2}\right]; F; n+l\right) - \rho\left(2\left[\frac{\overline{u}n}{2}\right]; F; n\right) \right| \leq \epsilon$$

for $0 \leq l \leq [\delta n]$, $n \geq n_0$.

Using this proposition and Jensen's inequality we can prove that the measure σ from Proposition 4.8 is the unit mass on $f_3(u)$ and since $b(f_3(u)) = a_3(u)$ this implies convergence to local equilibrium (see Rost, 1981, Section 4 for the details). $\qquad\Box$

4.3. *Multi-type models out of equilibrium.*

Proof of Theorem 3.9: First we want to outline the proof for convergence of $\frac{X^{(i)}(t)}{t}$ in distribution (as before we have $t \in \mathbb{R}_+$ or $t \in \mathbb{N}$ depending on the model). This follows the ideas from Ferrari and Kipnis (1995). We want to couple two TASEPs with initial configurations

$$\eta_0^1(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x \geq 1 \end{cases} \text{ and } \eta_0^2(x) = \begin{cases} 1 & x \leq -1 \\ 0 & x \geq 0 \end{cases}$$

in two different ways and calculate the difference $\mathbb{E}\left[S^1([rt], t)\right] - \mathbb{E}\left[S^2([rt], t)\right]$ in both couplings. $S^1([rt], t)$ and $S^2([rt], t)$ are the number of particles to the right of $[rt]$ at time t in η^1 and η^2. Using basic coupling (i.e. using the same Poisson processes $\{(P_t^x)_{t \geq 0} : x \in \mathbb{Z}\}$ or Bernoulli processes $\{(B_n^x)_{n \in \mathbb{N}} : x \in \mathbb{Z}\}$ for η^1 and η^2) gives

$$\mathbb{E}\left[S^1([rt], t)\right] - \mathbb{E}\left[S^2([rt], t)\right] = \mathbb{P}\left[X^{(i)}(t) > [rt]\right] \qquad (4.3)$$

since we can interpret the discrepancy between η^1 and η^2 as second class particle (see Section 2.3). This works for all models R0, R1, R2 and R3. The second coupling we want to use is called particle-particle coupling. We label the particles in η_0^1 and η_0^2 from right to left and let particles with the same label jump at the same time. Then under this coupling

$$\mathbb{E}\left[S^1([rt],t)\right] - \mathbb{E}\left[S^2([rt],t)\right] = \mathbb{P}\left[\eta_t^1([rt]+1) = 1\right] \tag{4.4}$$

in the models R0, R1 and R2. By Theorem 3.6 the right hand side of (4.4) converges to $f_i(r)$, so together with (4.3) we have

$$\mathbb{P}\left[X^{(i)}(t) > [rt]\right] \xrightarrow[t\to\infty]{} f_i(r)$$

which proves convergence in distribution for $i = 0, 1, 2$. However, in the model R3 we cannot use the particle-particle coupling as before because this is no longer a real coupling. If we let particles with the same label jump at the same time in η^1 and η^2 then the dynamics of the η^2 process are different from the η^1 process: In η^2 we update odd sites first and then even sites. We denote by $\mathbb{E}_{PP}$ the expectation in η^1 and η^2 if particles with the same label jump at the same time in η^1 and η^2 where we update in such a way that η^1 is still a TASEP with update rule R3. Then we have

$$\mathbb{E}_{PP}\left[S^1([rt],t)\right] = \mathbb{E}\left[S^1([rt],t)\right]$$

Notice that starting the second process with updating even sites does not change anything as there is no particle on an even site with an adjacent empty site in the initial configuration. If we remove the last update of even sites at time t this changes the value of $S^2([rt],t)$ if there is a jump from site $[rt]$ to site $[rt]+1$ during this update. But there can only be a jump from $[rt]$ to $[rt]+1$ while updating the even sites if $[rt]$ is even. Let us therefore consider odd sites $2[\frac{rt}{2}]+1$ first. We get

$$\mathbb{E}_{PP}\left[S^2\left(2\left[\frac{rt}{2}\right]+1,t\right)\right] = \mathbb{E}\left[S^2\left(2\left[\frac{rt}{2}\right]+1,t\right)\right] \tag{4.5}$$

and we still have

$$\mathbb{E}_{PP}\left[S^1\left(2\left[\frac{rt}{2}\right]+1,t\right)\right] - \mathbb{E}_{PP}\left[S^2\left(2\left[\frac{rt}{2}\right]+1,t\right)\right] = \mathbb{P}\left[\eta_t^1\left(2\left[\frac{rt}{2}\right]+2\right)=1\right]$$

as before. Hence we get

$$\mathbb{P}\left[X^{(3)}(t) > 2\left[\frac{rt}{2}\right]+1\right] = \mathbb{E}\left[S^1\left(2\left[\frac{rt}{2}\right]+1,t\right)\right] - \mathbb{E}\left[S^2\left(2\left[\frac{rt}{2}\right]+1,t\right)\right]$$

$$= \mathbb{P}\left[\eta_t^1\left(2\left[\frac{rt}{2}\right]+2\right)=1\right]$$

$$\xrightarrow[t\to\infty]{} a_3(r)$$

by the convergence to local equilibrium (Theorem 3.6). But by monotonicity we have

$$\mathbb{P}\left[X^{(3)}(t) > 2\left[\frac{rt}{2}\right]-1\right] \geq \mathbb{P}\left[X^{(3)}(t) > 2\left[\frac{rt}{2}\right]\right] \geq \mathbb{P}\left[X^{(3)}(t) > 2\left[\frac{rt}{2}\right]+1\right]$$

Hence

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{X^{(3)}(t)}{t} > r\right] = a_3(r)$$
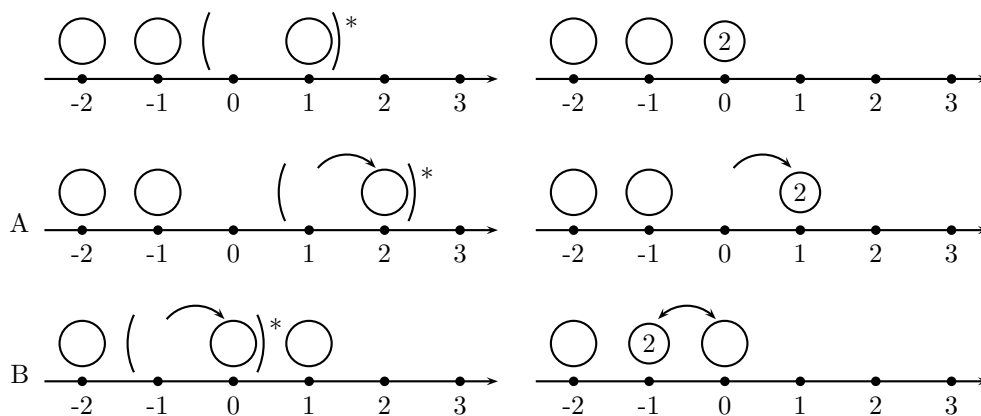
FIGURE 4.7. Pair representation of the second class particle: The
figures on the left show the system with the pair; the
figures on the right show the corresponding multi-type
system

Remark 4.10. If the second class particle starts on an odd site (with first class
particles to the left and holes/third class particles to the right) then

$$\frac{\widetilde{X}^{(3)}(t)}{t} \xrightarrow[t\to\infty]{a.s.} \widetilde{U}^{(3)}$$

and $\widetilde{U}^{(3)}$ has distribution function $\frac{a_3(1-\beta)}{1-a_3\beta}$ accordingly.

Now we want to prove almost sure convergence of the speed of a second class
particle. Our methods follow the approach in Ferrari and Pimentel (2005). The
idea is to establish a connection between the path of the second class particle and
a competition interface in the corresponding growth model. The cluster in the
growth model can be divided into two clusters corresponding to events happening
to the right and to the left of the second class particle. The interface between
these two clusters is called the competition interface. Using results about semi-
infinite geodesics it can be shown that this competition interface has almost surely
an asymptotic direction. This can be used to deduce that the second class particle
has almost surely an asymptotic speed and since we know the distribution of this
speed we can also calculate the distribution of the random angle of the competition
interface. In the following we will describe this method first for the TASEP in
continuous time, as given in Ferrari and Pimentel (2005), and then explain the
adjustments that have to be made in the TASEPs in discrete time.
In order to establish the connection between the second class particle and the
competition interface we represent the second class particle as a pair consisting of a
hole and a particle. This reduces our multi-type model to a model consisting only of
particles and holes and allows us to use the connection to last-passage percolation
developed in Section 2.2. We let the pair move as follows: If the particle of the pair
jumps to the right the pair moves to the right (A) and if a particle jumps from the
left into the hole of the pair the pair moves to the left (B), see Figure 4.7. Then
the pair behaves indeed like a second class particle. If we label the particles from
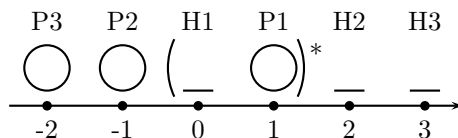
FIGURE 4.8. Pair representation of the second class particle with particles labelled from right to left and holes labelled from left to right

right to left and the holes from left to right as in Figure 4.8 then we can consider the process $(\varphi_n)_{n\geq 0}$ giving the labels of the pair after the nth jump involving the pair. We have $\varphi_0 = (1,1)$, as initially the pair consists of hole 1 and particle 1, and $\varphi_{n+1} - \varphi_n \in \{(0,1),(1,0)\}$. φ_n satisfies the recursion formula

$$\varphi_{n+1} = \begin{cases} \varphi_n + (1,0) & T(\varphi_n + (1,0)) < T(\varphi_n + (0,1)) \\ \varphi_n + (0,1) & T(\varphi_n + (1,0)) > T(\varphi_n + (0,1)) \end{cases} \tag{4.6}$$

If the first label increases the second class particle moves one step to the right and if the second label increases the second class particle moves one step to the left.

Remark 4.11. Note that by inserting an extra site we changed the parity of the sites to the right of and including the particle of the pair: The first hole to the right of the second class particle is at an odd site while the first hole to the right of the pair is at an even site. The second class particle itself is at an even site while the particle in the pair is at an odd site. This will be important for model R3.

Now we want to define geodesics in the last-passage percolation model to define a competition interface in the growth model. For $z, z' \in \mathbb{Z}_+^2$ the heaviest increasing path from z to z' (i.e. the path that achieves the maximum in $R(z, z')$) is called the geodesic from z to z'. Note that in the model in continuous time geodesics are unique. (In the models in discrete time we will need a rule to break ties to achieve uniqueness of the geodesics). A semi-infinite geodesic starting at z is a path $\pi = (z, z_1, z_2, \ldots)$ in $\mathbb{Z}_+^2$ such that for every $z' = z_k, z'' = z_l \in \pi$ the geodesic from z' to z'' is $(z_k, z_{k+1}, z_{k+2}, \ldots, z_l) \subset \pi$. For $\alpha \in [0, 90°]$ a α-geodesic is a semi-infinte geodesic with direction α. Now we colour every block $Q(i,j) = (i-1, i] \times (j-1, j]$ in $(\mathbb{R}_+)^2\backslash[0,1]^2$ either red if the geodesic from $(1,1)$ to (i,j) passes through $(1,2)$ or blue if it passes through $(2,1)$. The interface between these clusters is called the competition interface and an induction argument together with the recursion (4.6) shows that it is given by the process φ_n, see Proposition 3 in Ferrari and Pimentel (2005). Using results about the existence and uniqueness of α-geodesics (Propositions 7,8 and 9 in Ferrari and Pimentel, 2005) it can be shown that φ_n has almost surely an asymptotic direction and we can conclude that the second class particle has almost surely an asymptotic speed (Propositions 4 and 5 in Ferrari and Pimentel, 2005). Now we want to apply these methods to the discrete time TASEPs R1, R2 and R3.

R1. The last-passage percolation model for rule R1 was described in Section 2.2, and in particular just after (2.3). Now to adapt to the initial configuration

$$
\eta_0(x) = \begin{cases} 1 & x \le -1 \\ 0 & x = 0 \\ 1 & x = 1 \\ 0 & x \ge 2 \end{cases}
$$

we have to remove the '+1' weight from the edge between $(1,1)$ and $(2,1)$ and the weight from the vertex $(1,1)$ as in the initial configuration we are considering particle 1 has already jumped over hole 1. We colour $Q(1,2)$ red, $Q(2,1)$ blue and every other block $Q(i,j)$ in $(\mathbb{R}_+)^2 \setminus [0,1]^2$ either red if

$$
\widetilde{R}^{(1)}((1,2),(i,j)) > \widetilde{R}^{(1)}((2,1),(i,j))
$$

and blue if

$$
\widetilde{R}^{(1)}((1,2),(i,j)) \le \widetilde{R}^{(1)}((2,1),(i,j))
$$

(recall the defintion of R in (2.2); $\widetilde{R}^{(1)}$ is the corresponding quantity in model R1 with the changes mentioned above). This implies that if $Q(i,j+1)$ is red and $Q(i+1,j)$ is blue, then $Q(i+1,j+1)$ is red iff

$$
\widetilde{T}^{(1)}(i,j+1) \ge \widetilde{T}^{(1)}(i+1,j) \tag{4.7}
$$

and blue iff

$$
\widetilde{T}^{(1)}(i,j+1) < \widetilde{T}^{(1)}(i+1,j) \tag{4.8}
$$

where $\widetilde{T}^{(1)}$ is defined as in (2.3) but now in the model R1 with the modifications described above. The line $\varphi_n^{(1)}$ separating the two clusters is again called the competition interface and due to the way we defined the red and blue cluster we have again that the competition interface corresponds to the path of the second class particle. We can rewrite (4.7) and (4.8) in terms of $\varphi_n^{(1)}$ as

$$
\varphi_{n+1}^{(1)} = \begin{cases} \varphi_n^{(1)} + (1,0) & \widetilde{T}^{(1)}(\varphi_n^{(1)} + (1,0)) \le \widetilde{T}^{(1)}(\varphi_n^{(1)} + (0,1)) \\ \varphi_n^{(1)} + (0,1) & \widetilde{T}^{(1)}(\varphi_n^{(1)} + (1,0)) > \widetilde{T}^{(1)}(\varphi_n^{(1)} + (0,1)) \end{cases} \tag{4.9}
$$

Note the similarity of (4.6) with (4.9); the difference comes from the fact that in the models in discrete time ties are possible. As in Ferrari and Pimentel (2005) we want to prove that this competition interface has almost surely an asymptotic direction. First we note that the results in Ferrari and Pimentel (2005) about geodesics still hold with geometric weights attached to the vertices instead of exponential weights. Secondly, the results still hold in a model where '+1' weights are attached to *every* horizontal edge in $\mathbb{Z}_+^2$ as these weights do not affect the geodesics (they just give a constant weight to every path from z to z', $z, z' \in \mathbb{Z}_+^2$). The only difference in our case is that there is no weight attached to the edge from $(1,1)$ to $(2,1)$. But this local change does not affect the almost sure statements in Propositions 7,8 and 9 in Ferrari and Pimentel (2005). We conclude that the competition interface in our model has almost sure an asymptotic direction and it follows from arguments analogous to the ones in Ferrari and Pimentel (2005) that the speed of the second class particle converges almost surely.

R2. The result for model R2 follows again from the symmetry between R1 and R2.

R3. For the purpose of this section it is convenient to change the last-passage percolation model corresponding to model R3 a little bit. Instead of updating the even sites and then the odd sites during a single time-step, we separate the two batches of updates by a half time-step. Then the percolation model corresponding to the initial configuration

$$\eta_0(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x \geq 1 \end{cases}$$

has no weights attached to the edges, while the weights at the vertices are geometric with an extra $\frac{1}{2}$ added. As noticed in the beginning of this section, introducing an extra site into this model changes the parity of some sites. In order to deal with this we attach a single '$+\frac{1}{2}$' weight to the edge from $(1,1)$ to $(1,2)$ in the percolation model corresponding to the initial configuration

$$\eta_0(x) = \begin{cases} 1 & x \leq -1 \\ 0 & x = 0 \\ 1 & x = 1 \\ 0 & x \geq 2 \end{cases}$$

(where we also remove the weight from the vertex $(1,1)$). This ensures that we apply even/odd updates to the left of the particle of the pair and odd/even updates to the right. Then the movement of the pair $\varphi_n^{(3)}$ corresponds to the movement of the second class particle in a model with even/odd updates.

Again we colour $Q(1,2)$ red, $Q(2,1)$ blue and now every other block $Q(i,j)$ in $(\mathbb{R}_+)^2 \setminus [0,1]^2$ either red if

$$\widetilde{R}^{(3)}((1,2),(i,j)) \geq \widetilde{R}^{(3)}((2,1),(i,j)) \tag{4.10}$$

and blue if

$$\widetilde{R}^{(3)}((1,2),(i,j)) < \widetilde{R}^{(3)}((2,1),(i,j)) \tag{4.11}$$

The interface between these clusters is again the competition interface and is given by $\varphi_n^{(3)}$. In terms of $\varphi_n^{(3)}$ we get from (4.10) and (4.11) that

$$\varphi_{n+1}^{(3)} = \begin{cases} \varphi_n^{(3)} + (1,0) & \widetilde{T}^{(3)}(\varphi_n^{(3)} + (1,0)) < \widetilde{T}^{(3)}(\varphi_n^{(3)} + (0,1)) \\ \varphi_n^{(3)} + (0,1) & \widetilde{T}^{(3)}(\varphi_n^{(3)} + (1,0)) > \widetilde{T}^{(3)}(\varphi_n^{(3)} + (0,1)) \end{cases} \tag{4.12}$$

$\widetilde{R}^{(3)}$ and $\widetilde{T}^{(3)}$ are defined as in (2.2) and (2.5) but now we are considering the model R3 with the modifications described above. Due to the additional '$+\frac{1}{2}$' weight on the edge from $(1,1)$ to $(1,2)$ the competition interface never encounters any ties in this model, i.e. $\widetilde{T}^{(3)}(\varphi_n^{(3)} + (1,0)) = \widetilde{T}^{(3)}(\varphi_n^{(3)} + (0,1))$ does not occur. As in R1, the local change given by the extra '$+\frac{1}{2}$' edge-weight in this model does not change the almost sure statements in Propositions 7,8 and 9 in Ferrari and Pimentel (2005). The rest of the argument is the same as for model R1.

$\square$

Remark 4.12. We can use the known distributions of the speeds of the second class particles (see Theorem 3.9) together with the hydrodynamic limit results (see Theorem 3.6 and Remark 3.7) to prove the interesting result mentioned in Remark 3.10 that the distributions of the asymptotic direction of the competition interfaces in the models R1 and R3 are the same:

Proof: The proof exploits the connections made in Proposition 5 in Ferrari and Pimentel (2005). Similar to Ferrari and Pimentel (2005) we let $\psi_t^{(1)} = (I^{(1)}(t), J^{(1)}(t))$, $\psi_t^{(3)} = (I^{(3)}(t), J^{(3)}(t))$ be the position of the competition interface (i.e. the labels of the pair) at time t and denote by $\theta^{(1)}, \theta^{(3)} \in [0, 90°]$ the random angle of the competition interface for model R1 and R3, i.e.

$$\lim_{t \to \infty} \frac{\psi_t^{(i)}}{\left|\psi_t^{(i)}\right|} = e^{i\theta^{(i)}} = \left(\cos(\theta^{(i)}), \sin(\theta^{(i)})\right) \text{ for } i = 1, 3$$

By the arguments in the previous sections we know that these limits exist almost surely. Using the asymptotic shape of the growth models given in Remark 3.7 we also have

$$\lim_{t \to \infty} \frac{\psi_t^{(i)}}{t} = j^{(i)}(\theta^{(i)})e^{i\theta^{(i)}} \text{ a.s. for } i = 1, 3$$

where $j^{(i)}(\theta^{(i)})$ is the distance from the origin to the intersection of the line given by $\{(u, v) \in \mathbb{R}_+^2 : \tan(\theta^{(i)}) = \frac{v}{u}\}$ and the asymptotic *growth interface* $(x, g_i(x))$ for $i = 1, 3$. With the formulas for g_1 and g_3 in (3.1) and (3.3) we can calculate $j^{(1)}(\theta^{(1)})$ and $j^{(3)}(\theta^{(3)})$ explicitly:

$$j^{(1)}(\theta^{(1)}) = \frac{\beta}{\left(\sqrt{(1-\beta)\sin(\theta^{(1)})} + \sqrt{\cos(\theta^{(1)})}\right)^2}$$

and

$$j^{(3)}(\theta^{(3)}) = \frac{2\beta(2-\beta)}{\left((2-\beta)\sqrt{\sin(\theta^{(3)})} + 2\sqrt{(1-\beta)\cos(\theta^{(3)})}\right)^2 + \beta^2\cos(\theta^{(3)})}$$

By the connection between the path of the second class particle and the competition interface $(X^{(i)}(t) = I^{(i)}(t) - J^{(i)}(t), i = 1, 3,$ since the second class particle moves to the right iff the first label of the pair increases and to the left iff the second label of the pair increases) it follows that

$$\lim_{t \to \infty} \frac{X^{(i)}(t)}{t} = \lim_{t \to \infty} \frac{I^{(i)}(t) - J^{(i)}(t)}{t} = j^{(i)}(\theta^{(i)})\left(\cos(\theta^{(i)}) - \sin(\theta^{(i)})\right) \overset{\text{def}}{=} l_i(\theta^{(i)})$$

almost surely for $i = 1, 3$. Using the known distributions of the speed of the second class particle in model R1 and R3 (see Theorem 3.9) we can calculate

$$\mathbb{P}\left[\theta^{(1)} \le \alpha\right] = \mathbb{P}\left[l_1(\theta^{(1)}) \ge l_1(\alpha)\right] = f_1(l_1(\alpha))$$

and

$$\mathbb{P}\left[\theta^{(3)} \le \alpha\right] = \mathbb{P}\left[l_3(\theta^{(3)}) \ge l_3(\alpha)\right] = a_3(l_3(\alpha))$$

A calculation shows that

$$f_1(l_1(\alpha)) = a_3(l_3(\alpha))$$

$\square$

Proof of Theorem 3.13: Let the operators σ_n be defined by

$$\sigma_n Y = \begin{cases} \tau_n Y & Y_n < Y_{n+1} \\ Y & \text{otherwise} \end{cases}$$
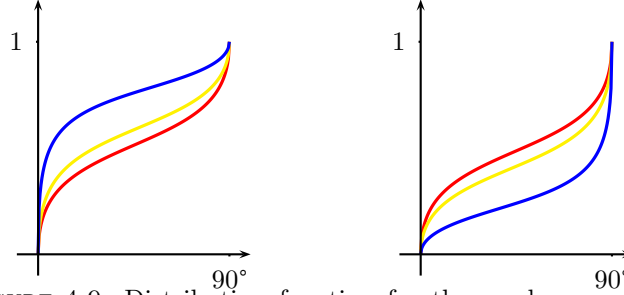
FIGURE 4.9. Distribution function for the random angle of the competition interface in the models R1 and R3 (left) and R2 (right) for $\beta = 0.1$ (red), $\beta = 0.5$ (yellow) and $\beta = 0.9$ (blue)

where τ_n exchanges Y_n and Y_{n+1} in Y. The proof of the following general Lemma is the same as in Amir et al. (2011) (Lemma 3.1).

Lemma 4.13. *For a fixed sequence $i_1, \ldots, i_k$ in $\mathbb{Z}$ we have*

$$\sigma_{i_k} \cdots \sigma_{i_1} \stackrel{d}{=} (\sigma_{i_1} \cdots \sigma_{i_k})^{-1}$$

We have the relation $Y_{X_n(m)}(m) = n$ between the X and the Y process. Since $\beta < 1$ each site has a positive probability that no jump occurs at that site at any given time. At each time-step these sites separate $\mathbb{Z}$ into finite intervals and the events on these intervals during that time-step are independent. In the model with updates from right to left we apply a finite sequence of operators $\sigma_{i_1} \cdots \sigma_{i_k}$ where $i_1, \ldots, i_k$ is an increasing sequence (since we update from right to left). Lemma 4.13 states that applying $\sigma_{i_1} \cdots \sigma_{i_k}$ is the same (in distribution) as applying $\sigma_{i_k} \cdots \sigma_{i_1}$ (i.e. updating from left to right) and taking the inverse permutation. But given the configuration $Y_n(0)$ and performing the updates from left to right we get $Y_n^{(2)}(1)$ and this is the inverse permutation of $X_n^{(2)}(1)$. So

$$Y_n^{(1)}(1) \stackrel{d}{=} X_n^{(2)}(1)$$

Inductively we get that this holds for all $m > 0$. The other three parts follow in the same way. In the case with even/odd updates, $i_1, \ldots, i_k$ is a sequence such that there exists a $1 \leq j \leq k+1$ such that i_l is odd for $l < j$ and i_l is even for $l \geq j$. □

In order to prove Theorem 3.12 we will need the following Lemma. We state it here for the model R1, but analogous results hold for the other models as well. The Lemma corresponds to Lemma 4.1 in Amir et al. (2011).

Lemma 4.14. *Consider two TASEPs, $Y^{(1)}$ and $\widetilde{Y}^{(1)}$, as functions of the same Bernoulli points on $\mathbb{Z} \times \mathbb{N}$ (i.e. under basic coupling). We set $Y_n^{(1)}(0) = n$ and $\widetilde{Y}_n^{(1)}(0) = \sigma_j \cdots \sigma_{j+k} Y_n^{(1)}(0)$ for some $j \in \mathbb{Z}$ and $k \geq 0$. Let $\{U_n^{(1)}\}$ be the speed process of $Y^{(1)}$ and $\{\widetilde{U}_n^{(1)}\}$ be the speed process of $\widetilde{Y}^{(1)}$. Then $\widetilde{U}^{(1)} = \sigma_{j+k} \cdots \sigma_j U^{(1)}$.*

Proof: Every other particle than $\{j, \ldots j+k+1\}$ is either stronger than all particles $\{j, \ldots, j+k+1\}$ or weaker than all particles $\{j, \ldots, j+k+1\}$. Any swap of a particle other than $\{j, \ldots, j+k+1\}$ will happen in both $Y^{(1)}$ and $\widetilde{Y}^{(1)}$. So for any $i \notin \{j, \ldots, j+k+1\}$ we have $X_i^{(1)}(m) = \widetilde{X}_i^{(1)}(m)$ for all $m \geq 0$ and therefore $U_i^{(1)} = \widetilde{U}_i^{(1)}$ for those i. In $\widetilde{Y}^{(1)}$ particle $j+k+1$ is always to the left of all other

particles $\{j, \ldots, j+k\}$. So $\widetilde{U}_{j+k+1}^{(1)} = \min\{U_j^{(1)} \ldots U_{j+k+1}^{(1)}\}$. Define $j \leq r \leq j+k+1$ by

$$\min\left\{U_j^{(1)} \ldots U_{j+k+1}^{(1)}\right\} = U_r^{(1)}$$

Then for $i = r, r+1, \ldots, j+k$ we have $\widetilde{U}_i^{(1)} = U_{i+1}^{(1)}$ and for $i = j, j+1, \ldots, r-1$

$$\widetilde{U}_j^{(1)} = \max\left\{U_j^{(1)}, U_{j+1}^{(1)}\right\}$$

$$\widetilde{U}_{j+1}^{(1)} = \max\left\{\min\left\{U_j^{(1)}, U_{j+1}^{(1)}\right\}, U_{j+2}^{(1)}\right\}$$

$$\widetilde{U}_{j+2}^{(1)} = \max\left\{\min\left\{\min\left\{U_j^{(1)}, U_{j+1}^{(1)}\right\}, U_{j+2}^{(1)}\right\}, U_{j+3}^{(1)}\right\}$$

$$\ldots$$

This shows that $\widetilde{U}^{(1)} = \sigma_{j+k} \cdots \sigma_j U^{(1)}$. $\hfill\square$

Proof of Theorem 3.12: Consider a Bernoulli process on $\mathbb{Z} \times \mathbb{Z}$. Half of this process $(\mathbb{Z} \times \mathbb{N})$ is used to construct the TASEP $Y^{(1)}$. For any $l \in \mathbb{Z}$ we can translate the Bernoulli process by l (i.e. take points of the form $(n, m+l)$ where (n, m) is in the original process). We can restrict this translated process to $\mathbb{Z} \times \mathbb{N}$ and use this restricted process to construct another TASEP. Let $U^{(1)}(l) = \{U_n^{(1)}(l)\}$ be the speed process for the TASEP that has been constructed using the Bernoulli process translated by l. For every l, $U^{(1)}(l)$ has distribution $\mu^{(1)}$. So we have to show that $\{U_n^{(1)}(l)\}$ behaves like a TASEP with updates from left to right. In order to do this we look at a transition $\{U_n^{(1)}(l)\} \to \{U_n^{(1)}(l+1)\}$. The effect on the original TASEP of changing from translating by l to translating by $l+1$ is that some finite sequences of σ operators of the form $\sigma_j \cdots \sigma_{j+k}$ are added to be applied to the TASEP before the original sequence of operations. At each location a σ operator is added with probability β. The previous Theorem shows, that applying each of these finite sequences has the same effect on the speeds as applying each sequence in reverse order to the speed process. This shows that $\{U_n^{(1)}(l)\}$ behaves like a TASEP with updates from left to right and therefore the measure $\mu^{(1)}$ is stationary for the TASEP with updates from left to right.

The proofs for the other three models are essentially the same (using the appropriate versions of Lemma 4.14). $\hfill\square$

The following Lemma will allow us to do the explicit calculations for the joint densities of the speeds in Theorems 3.15 - 3.19 using the connection between queueing models and the invariant measures introduced in Theorem 3.3. Here $D(g_i)$ is the domain of the distribution function g_i $(i = 0, 1, 2, 3, 4)$.

Lemma 4.15. *If $F : D(g_i) \to \{1, \ldots, N\}$ is non-decreasing then for the TASEP $\{Y_n^{(i)}(m)\}_{n,m}$ the distribution of $\{F(U_n^{(i)})\}_n$ is the unique ergodic stationary measure of the multi-type TASEP model $Rj(i)$ with types $\{1, \ldots, N\}$ and densities $\lambda_l = g_i(\sup\{F^{-1}(l)\}) - g_i(\inf\{F^{-1}(l)\})$ for type $l = 1, \ldots, N$ (j as in Theorem 3.12).*

Proof: The proof is analogous to the proof of Corollary 5.4 in Amir et al. (2011). $\hfill\square$

With the help of this Lemma we can do all the calculations needed for the results in Theorems 3.15 - 3.19. Depending on the model we are considering we will choose the function F from Lemma 4.15 to be $F_i = \min\{j : g_i(u) < x_j\}$ for some increasing

sequence $(x_1, \ldots, x_{N-1})$ in $[0, 1]$. Then we put $V_n = F(U_n)$ and the distribution of the Vs is given by the invariant measure for the multi-type models and can be calculated explicitly using the queueing representation.

Proof of Theorem 3.15: By Lemma 4.15 we have with $N = 3$ that $F_1(U_n)$ is distributed according to the unique ergodic stationary measure of a 3-type TASEP with updates from left to right and densities

$$\lambda_1 = \mathbb{P}\left[x_0 < g_1(U_n) < x_1\right] = x_1 - x_0 = x_1$$
$$\lambda_2 = \mathbb{P}\left[x_1 < g_1(U_n) < x_2\right] = x_2 - x_1$$
$$\lambda_3 = \mathbb{P}\left[x_2 < g_1(U_n) < x_3\right] = x_3 - x_2 = 1 - x_2$$

λ_1 is the density of first class particles, λ_2 is the density of second class particles and λ_3 is the density of third class particles (or holes). Recall that $V_n = F_1(U_n)$. Using the queueing representation for the unique ergodic stationary measure of a 3-type TASEP we can calculate the joint distribution (V_0, V_1) explicitly. This distribution depends on the x_i. Taking suitable derivatives with respect to these x_i we get the density of the corresponding speeds. We have for example

$$\mathbb{P}\left[U_0 < g_1^{-1}(x_1) < U_1 < g_1^{-1}(x_2)\right] = \mathbb{P}\left[V_0 = 1, V_1 = 2\right]$$
$$= x_1 x_2 \left(x_2 - x_1\right)$$

since the probability of having a second class particle at position 1 is $x_2 - x_1$ (since this is the density of second class particles) and to have a first class particle at position 0 we then have to have an arrival (probability x_1) and a service (probability x_2) because having a second class particle at site 1 means that the queue was empty at that time (so in order to have a departure at site 0 we need an arrival at site 0). Remember that in Theorem 3.3 the particles in the TASEP jumped from the right to the left. If we want to consider the TASEP with jumps from the left to the right (and that is what we are doing here) we have to read the queues from right to left. So position 1 comes before position 0 and the probability of having a second class particle at position 1 is independent of arrivals and services at position 0. So for $u_0 < u_1$ we put $x_1 = g_1(u_0)$ and $x_2 = g_1(u_1)$ and get as density

$$\mathbb{P}\left[U_0 \in du_0, U_1 \in du_1\right] = \frac{dx_1}{du_0} \frac{dx_2}{du_1} \frac{d}{dx_1} \frac{d}{dx_2} x_1 x_2 \left(x_2 - x_1\right)$$
$$= \frac{1 - \beta}{4\beta^2} \left(1 - u_0\right)^{-\frac{3}{2}} \left(1 - u_1\right)^{-\frac{3}{2}} \left(2g_1(u_1) - 2g_1(u_0)\right)$$
$$= \frac{1 - \beta}{2\beta^3} \left(1 - u_0\right)^{-\frac{3}{2}} \left(1 - u_1\right)^{-\frac{3}{2}}$$
$$\cdot \left(\sqrt{\frac{1 - \beta}{1 - u_1}} - \sqrt{\frac{1 - \beta}{1 - u_0}}\right)$$

Similarly, we have

$$\mathbb{P}\left[g_1^{-1}(x_1) < U_1 < g_1^{-1}(x_2) < U_0\right] = \mathbb{P}\left[V_0 = 3, V_1 = 2\right]$$
$$= \left(1 - x_2\right) \left(x_2 - x_1\right)$$

and therefore we get as density for $u_0 > u_1$ (putting $x_1 = g_1(u_1)$, $x_2 = g_1(u_0)$)

$$
\begin{aligned}
\mathbb{P}\left[U_0 \in du_0, U_1 \in du_1\right] &= \left(-\frac{dx_1}{du_0}\right)\left(-\frac{dx_2}{du_1}\right)\frac{d}{dx_1}\frac{d}{dx_2}\left(1-x_2\right)\left(x_2-x_1\right) \\
&= \frac{1-\beta}{4\beta^2}\left(1-u_0\right)^{-\frac{3}{2}}\left(1-u_1\right)^{-\frac{3}{2}} \\
&= g_1'(u_0)g_1'(u_1)
\end{aligned}
$$

To get the density for $u_0 = u_1$ we consider

$$
\begin{aligned}
\mathbb{P}\left[g_1^{-1}(x_1) < U_0, U_1 < g_1^{-1}(x_2)\right] &= \mathbb{P}\left[V_0 = 2, V_1 = 2\right] \\
&= \left(1-x_1\right)x_2\left(x_2-x_1\right)
\end{aligned}
$$

and let $x_1, x_2 \to g_1(u)$. We get

$$
\begin{aligned}
\mathbb{P}\left[U_0, U_1 \in du\right] &= \lim_{x_1, x_2 \to g_1(u)} \frac{\left(1-x_1\right)x_2\left(x_2-x_1\right)}{g_1^{-1}(x_2) - g_1^{-1}(x_1)} \\
&= \frac{\sqrt{1-\beta}\left(1-g_1(u)\right)g_1(u)}{2\beta\left(1-u\right)^{\frac{3}{2}}} \\
&= \frac{\sqrt{1-\beta}}{2\beta^2\left(1-u\right)^{\frac{3}{2}}}\left(1-\frac{1}{\beta}\right) + \frac{1-\beta}{2\beta^2\left(1-u\right)^2}\left(\frac{2}{\beta}-1\right) \\
&\quad - \frac{\sqrt{1-\beta}\left(1-\beta\right)}{2\beta^3\left(1-u\right)^{\frac{5}{2}}}
\end{aligned}
$$

To get the probabilities in the Theorem we only have to integrate the densities over the appropriate ranges of u_0, u_1 and u. Alternatively we can use the following:

$$
\begin{aligned}
\mathbb{P}\left[U_0^{(1)} < U_1^{(1)}\right] &= \mathbb{P}\left[g_1(U_0^{(1)}) < g_1(U_1^{(1)})\right] \\
&\overset{(*)}{=} \mathbb{P}\left[g_0(U_0^{(0)}) < g_0(U_1^{(0)})\right] \\
&= \mathbb{P}\left[U_0^{(0)} < U_1^{(0)}\right] \\
&= \frac{1}{3}
\end{aligned}
$$

$(*)$ follows from the fact that the distribution of $\{g_1(U_n^{(1)})\}$ is the unique translation invariant stationary ergodic measure for the TASEP R2 with marginals uniform on $[0,1]$. The distribution of $\{g_0(U_n^{(0)})\}$ is the unique translation invariant stationary ergodic measure for the TASEP in continuous time. Since the stationary distributions for the multi-type TASEPs R0 and R2 are the same (see Theorem 3.3) $\{g_1(U_n^{(1)})\}$ has the same distribution as $\{g_0(U_n^{(0)})\}$. $\square$

The proofs for Theorems 3.17 - 3.19 work in exactly the same way.

5. Fully parallel updates

Finally we mention the model with "fully parallel updates". If an update occurs at site x at time t (which happens with probability β as usual), this update causes a jump from x to $x+1$ if and only if $\eta_{t-1}(x) = 1$ and $\eta_{t-1}(x+1) = 0$ (that is, the jump is already possible before any other updates at the current time-step are performed).

There are several important differences between this model and the models we have studied earlier. The Bernoulli product measures ν_ρ are no longer invariant. Furthermore, the basic coupling no longer preserves an ordering between different initial configurations, and so it is no longer clear how to define a multi-class system. If we use basic coupling to couple two systems which start with one discrepancy at the origin then this single discrepancy can generate additional discrepancies. It would already be interesting to know how the leftmost and rightmost discrepancies behave. Do they have asymptotic speeds, and if so are the speeds random or deterministic? There is still a natural percolation representation, and we can still obtain a hydrodynamic limit result in the sense that $\frac{1}{n}\sum_{un<k<vn}\eta_t(n)$ converges a.s. to the constant value $\int_u^v f(w)dw$, for $u < v$ and some function f, but the stronger result that $\lim_{n\to\infty}\mathbb{E}\left[\eta_n(k)\right]$ exists and is equal to $f(u)$ whenever $\frac{k}{n}$ tends to u does not follow using the same methods as in the proof of Theorem 3.6.

Acknowledgments

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