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Large deviations for the largest eigenvalue of an Hermitian Brownian motion

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Abstract. We establish a large deviation principle for the process of the largest eigenvalue of an Hermitian Brownian motion. By a contraction principle, we recover the LDP for the largest eigenvalue of a rank one deformation of the GUE.

1. Introduction

The Gaussian unitary ensemble (GUE) is probably the most studied ensemble of random matrices. In this work, we will focus on a dynamical version of the GUE, introduced in 1962 by Dyson : he defined the Hermitian Brownian motion whose set of eigenvalues is a time-dependent Coulomb gas, consisting in particles evolving according to Brownian motions under the influence of their mutual electrostatic repulsions. More precisely, let $(\beta_{ij}, \beta'_{ij})_{1\leq i\leq j\leq N}$ be a collection of independent identically distributed standard real Brownian motions; the Hermitian Brownian motion $(H_N(t))_{t\geq 0}$ is the random process, taking values in the space of $N \times N$

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Hermitian matrices, with entries $(H_N)_{kl}$ given, for $k \leq l$ by

$$(H_N)_{kl} = \begin{cases} \frac{1}{\sqrt{2N}}(\beta_{kl} + i\beta'_{kl}), & \text{if } k < l, \\ \frac{1}{\sqrt{N}}\beta_{kk}, & \text{if } k = l. \end{cases}$$

Dyson (1962) (see also Guionnet (2009, 12.1)) showed that the eigenvalues of $(H_N(t))_{t\geq 0}$ satisfy the following system of stochastic differential equations (SDE)

$$d\lambda_i(t) = \frac{1}{\sqrt{N}} dB_i(t) + \frac{1}{N}\sum_{j\neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt,\ t \geq 0,\ i, j = 1, \ldots, N \tag{1.1}$$

where B_i are independent standard real Brownian motions.

It was rigorously shown in Cépa and Lépingle (1997) that this system of SDE admits a unique strong solution and the eigenvalues do not collide.

The process of the eigenvalues is called Dyson Brownian motion. Almost surely (a.s.), for any $t \geq 0$, the corresponding spectral measure $(\mu_N)_t := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i(t)}$ converges weakly to the semicircular distribution σ_t given by

$$d\sigma_t(x) = \frac{1}{2\pi t}\mathbf{1}_{[-2\sqrt{t}, 2\sqrt{t}]}\sqrt{4t - x^2}dx \quad \text{and} \quad \sigma_0 = \delta_0. \tag{1.2}$$

Let us now recall some large deviations results. For the global regime in the static case (GUE), a large deviation principle (LDP) in the scale N^2 was established in Ben Arous and Guionnet (1997) for the spectral measure of size N. There exists also a dynamical version of this result : Cabanal Duvillard and Guionnet (2001), Guionnet (2009) showed the following.

Let $C([0,1]; \mathcal{P}(\mathbb{R}))$, the set of continuous functions on $[0,1]$ with values in the set $\mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$. We equip this set with the metric $d(\mu, \nu) = \sup_{t\in[0,1]} d_{Lip}(\mu_t, \nu_t)$ where

$$d_{Lip}(\mu_t, \nu_t) = \sup_{f\in\mathcal{F}_{Lip}} \left| \int f d\mu_t - \int f d\nu_t \right|$$

where $\mathcal{F}_{Lip}$ denotes the space of bounded Lipschitz functions on $\mathbb{R}$ with Lipschitz and uniform bound less than 1. Then the process $((\mu_N)_t)_{0\leq t\leq 1}$ satisfies a LDP in the scale N^2 with respect to the topology inherited from the metric d.

We now define $H_N^\theta(t) = H_N(t) + H_N^\theta(0)$ the Hermitian Brownian motion starting from $H_N^\theta(0) := \mathrm{diag}(\theta, 0, \ldots, 0)$, with $\theta \geq 0$ and denote by $\lambda_1^{\theta,N}(t) \geq \lambda_2^{\theta,N}(t) \geq \ldots \geq \lambda_N^{\theta,N}(t)$ the set of ordered eigenvalues of $H_N^\theta(t)$. All the results stated above about the global regime of the Hermitian Brownian motion starting from 0 will stay valid for $(H_N^\theta(t))_{t\geq 0}$. In this work, we will be interested in the process of the maximal eigenvalue $(\lambda_1^{\theta,N}(t))_{t\geq 0}$, that is the largest particle of Dyson Brownian motion.

In the case $\theta = 0$, the corresponding quantity in the static case is just the maximal eigenvalue of the GUE. It is well known (see for example Bai and Yin (1988)) that it converges a.s. to 2. In the case $\theta > 0$ a similar result holds for a rank one additive deformations of the GUE (see for example Péché (2006)).

These results can be easily extended to the a.s. convergence of our process, in the topology of uniform convergence for continuous functions on $[0,1]$, towards the function $(f_\theta(t))_{t\geq 0}$ given by:

$$\left\{\begin{array}{ll} f_\theta(t) = 2\sqrt{t} & \text{if } \theta = 0, \\ f_\theta(t) = \left\{\begin{array}{ll} \theta + \frac{t}{\theta} & t \leq \theta^2 \\ 2\sqrt{t} & t \geq \theta^2 \end{array}\right. & \text{if } \theta > 0. \end{array}\right.$$

In particular, this result can be seen as a direct consequence of our main result stated below.

At the level of large deviations, the LDP for the largest eigenvalue of the GOE with a scale N was obtained in Ben Arous et al. (2001). This result was extended to a rank one perturbation of the GUE/GOE by one of the author in Maïda (2007). The main goal of this paper will be to prove a dynamical version of these two results. More precisely, we will consider the process $(\lambda_1^{\theta,N}(t))_{0\leq t\leq 1}$ as a sequence of random variables with values in the space $C_\theta([0,1],\mathbb{R})$ of continuous functions from $[0,1]$ to $\mathbb{R}$ equal to θ at zero and investigate its LDP in this space endowed with the uniform convergence. Our main result can be stated as follows.

Theorem 1.1. *The law of $(\lambda_1^{\theta,N}(t))_{0\leq t\leq 1}$ satisfies a large deviation principle on $C_\theta([0,1];\mathbb{R})$ equipped with the topology of uniform convergence, in the scale N, with good rate function*

$$I_\theta(\varphi) = \left\{\begin{array}{l} \frac{1}{2}\int_0^1 \left(\dot{\varphi}(s) - \frac{1}{2s}\left(\varphi(s) - \sqrt{\varphi(s)^2 - 4s}\right)\right)^2 ds, \\ \textit{if } \varphi \textit{ absolutely continuous and } \varphi(t) \geq 2\sqrt{t}\ \forall t \in [0,1], \\ +\infty, \ \textit{otherwise}. \end{array}\right. \tag{1.3}$$

As a consequence of our main result, we will recover by contraction the fixed-time large deviation principles already shown in Ben Arous et al. (2001) and Maïda (2007). The following result is a corrected version of Theorem 1.1 in Maïda (2007), the proof there is correct.

Theorem 1.2. *The largest eigenvalue of $H_N^\theta(1)$ satisfies a large deviation principle in the scale N, with good rate function K_θ defined as follows:*

- *If $\theta \leq 1$,*

$$K_\theta(x) = \left\{\begin{array}{ll} +\infty, & \textit{if } x < 2 \\ \displaystyle\int_2^x \sqrt{z^2-4}\, dz, & \textit{if } 2 \leq x \leq \theta + \frac{1}{\theta}, \\ M_\theta(x), & \textit{if } x \geq \theta + \frac{1}{\theta}, \end{array}\right.$$

$$\textit{with} \quad M_\theta(x) = \frac{1}{2}\int_2^x \sqrt{z^2-4}\, dz - \theta x + \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{2}\theta^2 + \log\theta.$$

- *If $\theta \geq 1$,*

$$K_\theta(x) = \left\{\begin{array}{ll} +\infty, & \textit{if } x < 2 \\ L_\theta(x), & \textit{if } x \geq 2, \end{array}\right.$$

$$\textit{with} \quad L_\theta(x) = \frac{1}{2}\int_{\theta+\frac{1}{\theta}}^x \sqrt{z^2-4} dz - \theta\left(x - \left(\theta + \frac{1}{\theta}\right)\right) + \frac{1}{4}\left(x^2 - \left(\theta + \frac{1}{\theta}\right)^2\right).$$

Before going further, let us make a few remarks :

Remark 1.3. (1) For the sake of simplicity the theorems above are stated and proven in the paper for the Hermitian Brownian motion but we want to mention that they can be easily extended to the symmetric Brownian motion. With the notations already introduced above, the latter is defined as the random process taking values in the space of $N \times N$ real symmetric matrices so that

$$(S_N)_{kl} = \frac{1}{\sqrt{N}}\beta_{kl}, \text{ if } k < l, \; (S_N)_{kk} = \sqrt{\frac{2}{N}}\beta_{kk}.$$

The process of its eigenvalues satisfies the following system of SDE

$$d\lambda_i(t) = \frac{\sqrt{2}}{\sqrt{N}} dB_i(t) + \frac{1}{N}\sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt, \; t \geq 0, \; i = 1, \ldots, N$$

and its law satisfies a LDP with good rate function simply given by $\frac{1}{2} I_\theta$.

The proof that will be developed in the sequel can be adapted to the symmetric case with a few minor changes left to the reader.

(2) In the whole introduction till now, we have considered our processes on the set of times $t \in [0, 1]$ for simplicity but everything could be easily generalised to any compact set $[0, T]$ for $T > 0$.

(3) In the sequel, we will specify the superscript θ, N in the statements but drop it in the proofs, unless there is any ambiguity.

Let us now specify a little bit the main features of the strategy of the proof. In both Ben Arous et al. (2001) and Maïda (2007), the fact that the deviations of the spectral measure and those of the largest eigenvalues do not occur in the same scale plays a crucial role and so will be in the proof of our result : in the scale at which we look at the largest eigenvalue, the spectral measure of all but the largest eigenvalue is already well concentrated around the semicircle law.

In the static case, the LDP was shown using the explicit expression of the joint distribution of the N eigenvalues. In the dynamical case, the proof relies on stochastic calculus using that the process of the eigenvalues satisfies the system of SDE (1.1). Roughly speaking, the largest eigenvalue is a solution of a SDE of the form

$$d\lambda_1(t) = \frac{1}{\sqrt{N}} dB(t) + b(\lambda_1(t), (\nu_N)_t) dt,$$

with B a standard real Brownian motion, $\nu_N := \frac{1}{N-1}\sum_{i=2}^N \delta_{\lambda_i(t)}$ the empirical distribution of all but the largest eigenvalues and the drift $b(x, \nu)$ to be explicited in the sequel. In the scale of interest, ν_N is close to σ and the rate function I_θ is the one predicted by the Freidlin-Wentzell Theorem (see Dembo and Zeitouni (2010, Th.5.6.3)) for the SDE

$$d\lambda_1(t) = \frac{1}{\sqrt{N}} dB(t) + b(\lambda_1(t), \sigma_t) dt.$$

One of the main difficulties will be to deal with the singularity of the drift b, as for some $x \in \mathbb{R}$, $\nu \mapsto b(x, \nu)$ is not a continuous function for the weak convergence of probabilities.

The organisation of the paper will be the following. To prove our main result, we first establish the exponential tightness of the process $(\lambda_1(t))_{0 \leq t \leq 1}$ stated in Proposition 2.1 and proved in Section 2. A short Section 3 will be devoted to the study of the rate function I_θ, where we check in particular its lower semicontinuity. Section 4 is devoted to the proof of the lower bound, stated in Proposition 4.1. The upper bound is given in (5.1) and obtained along Section 5. Then Theorem 1.1 will follow from the exponential tightness, the lower bound obtained in Proposition

4.1 and the weak upper bound (5.1) (see Dembo and Zeitouni (2010, Chapt. 4) or Anderson et al. (2010, Cor. D.6 and Th. D.4)). Finally, in Section 6, we recover by contraction principle the fixed-time LDP stated in Theorem 1.2.

2. Exponential tightness

We want to show the exponential tightness of the process $(\lambda_1(t))_{0\leq t\leq 1}$ in scale N that is

Proposition 2.1. *For all L, there exists N_0 and a compact set K_L of $C_\theta([0,1];\mathbb{R})$ such that:*

$$\forall N\geq N_0, \mathbb{P}(\lambda_1^{\theta,N}\notin K_L)\leq \exp(-LN).$$

From the description of the compact sets of $C([0,1];\mathbb{R})$ (Ascoli theorem), it is enough to show (see Revuz and Yor (1999, Chapter XIII, Section 1), Cabanal Duvillard and Guionnet (2001, Section 2.3)) the following lemma

Lemma 2.2. *For any $\eta>0$, there exists δ_0 such that for any $\delta<\delta_0$, for all N, $p\leq N$ and $s\in[0,1]$,*

$$\mathbb{P}\left(\sup_{s\leq t\leq s+\delta}|\lambda_p^{\theta,N}(t)-\lambda_p^{\theta,N}(s)|\geq\eta\right)\leq\exp\left(-\frac{1}{10}N\frac{\eta^2}{\delta}\right).$$

To get the proposition, for a fixed L, we choose $p=1$, any η and then δ small enough so that $\frac{\eta^2}{10\delta}>L$.

Proof of lemma 2.2: Let $0\leq s\leq 1$.

Let us denote by $\tilde{H}_N$ the Hermitian Brownian motion defined, for $u\geq 0$, by $\tilde{H}_N(u)=H_N^\theta(u+s)-H_N^\theta(s)$ and by $(\tilde{\lambda}_i(u))_{u\geq 0}$ its eigenvalues, in decreasing order. From a classical relation between eigenvalues (usually called Weyl's interlacing inequalities), for $t\geq s$,

$$\lambda_p^{\theta,N}(s)+\tilde{\lambda}_N(t-s)\leq\lambda_p^{\theta,N}(t)\leq\lambda_p^{\theta,N}(s)+\tilde{\lambda}_1(t-s)$$

so that

$$|\lambda_p^{\theta,N}(t)-\lambda_p^{\theta,N}(s)|\leq\max\left(\tilde{\lambda}_1(t-s),-\tilde{\lambda}_N(t-s)\right)=\|\tilde{H}_N(t-s)\|$$

where $\|.\|$ denotes the operator norm on matrices. For any $\eta>0$

$$\begin{aligned}\mathbb{P}\left(\sup_{s\leq t\leq s+\delta}|\lambda_p^{\theta,N}(t)-\lambda_p^{\theta,N}(s)|\geq\eta\right) &\leq \mathbb{P}\left(\sup_{s\leq t\leq s+\delta}\|\tilde{H}_N(t-s)\|\geq\eta\right)\\ &= \mathbb{P}\left(\sup_{0\leq u\leq\delta}\|\tilde{H}_N(u)\|\geq\eta\right)\\ &= \mathbb{P}\left(\sqrt{\delta}\sup_{0\leq u\leq 1}\|H_N^0(u)\|\geq\eta\right),\end{aligned}$$

where we used that $\tilde{H}_N$ has the same law as H_N^0 and the scaling invariance of this law. Therefore, it is enough to show that for M large enough and for all N,

$$\mathbb{P}\left(\sup_{0\leq u\leq 1}\|H_N^0(u)\|\geq M\right)\leq\exp(-NM^2/10). \tag{2.1}$$

Indeed, from Ben Arous et al. (2001), Lemma 6.3, we have: for y large enough, for all N,

$$\mathbb{P}(\|H_N^0(1)\| \geq y) \leq \exp(-Ny^2/9). \tag{2.2}$$

This implies that for $0 < a < 1/9$, $\mathbb{E}(e^{aN\|H_N^0(1)\|^2}) \leq 2e^{aNy_0^2}$ for some y_0 large enough and all N. Now, $(\|H_N^0(u)\|)_{0\leq u\leq 1}$ is a positive submartingale and from Doob's inequalities, all the moments of $\sup_{0\leq u\leq 1}\|H_N^0(u)\|$ are bounded by those of $\|H_N^0(1)\|$ (up to a constant 4). Therefore,

$$\mathbb{E}(e^{aN\sup_{0\leq u\leq 1}\|H_N^0(1)\|^2}) \leq 4\mathbb{E}(e^{aN\|H_N^0(1)\|^2}) \leq 8e^{aNy_0^2}.$$

From Markov's inequality,

$$\mathbb{E}(\sup_{0\leq u\leq 1}\|H_N^0(1)\| \geq M) \leq 8e^{-aNM^2}e^{aNy_0^2} \leq e^{-a'NM^2}$$

for $0 < a' < a$ and M large enough, proving (2.1). □

3. Some insight on the expected rate function

Before going into the proof of the lower bound, we gather hereafter some useful remarks about the function I_θ defined in Theorem 1.1. In particular, we show in this section that it is lower semi-continuous.

We introduce the following notations : for μ a probability measure on $\mathbb{R}$ and $x \in \mathbb{R}$, we define

$$b(x,\mu) = \int_{-\infty}^{x} \frac{d\mu(y)}{x-y} \in \mathbb{R}_+ \cup \{\infty\}. \tag{3.1}$$

For $\mu \in \mathcal{P}(\mathbb{R})$, we denote by $r(\mu)$ the right end-point of the support of μ. Let $(\varphi,\mu) \in C_\theta([0,1];\mathbb{R}) \times C([0,1];\mathcal{P}(\mathbb{R}))$ such that for all $t \in [0,1]$, $\varphi(t) > r(\mu_t)$. Then, $b(\varphi(t),\mu_t)$ is bounded. We set

$$\mathcal{H} := \{h \in C([0,1],\mathbb{R})/h \text{ absolutely continuous}, \dot{h} \in \mathbb{L}^2([0,1])\}$$

with $\mathbb{L}^2([0,1])$ the set of square-integrable functions from $[0,1]$ to $\mathbb{R}$ equipped with its usual $\mathbb{L}^2$-norm, denoted by $\|\cdot\|_2$. For any $h \in \mathcal{H}$ we define

$$\begin{aligned} G(\varphi,\mu;h) &= h(1)\varphi(1) - h(0)\varphi(0) - \int_0^1 \varphi(s)\dot{h}(s)ds - \int_0^1 b(\varphi(s),\mu_s)h(s)ds, \\ F(\varphi,\mu;h) &:= G(\varphi,\mu;h) - \frac{1}{2}\int_0^1 h^2(s)ds. \end{aligned} \tag{3.2}$$

For $\sigma := (\sigma_t)_{t\geq 0}$ the semicircular process defined in (1.2), the condition $\varphi(t) > r(\mu_t)$ reads $\varphi(t) > 2\sqrt{t}$ and one can check that $F(\varphi,\sigma;h)$ is also well defined under the weaker assumption that $\varphi(t) \geq 2\sqrt{t}$ for all $t \in [0,1]$. It is indeed well known (see for example Hiai and Petz (2000, p. 94)) that

$$b(\varphi(t),\sigma_t) = \frac{1}{2t}(\varphi(t) - \sqrt{\varphi^2(t) - 4t}), \tag{3.3}$$

so that $0 \leq b(\varphi(t),\sigma_t) \leq \frac{1}{\sqrt{t}}$ for $\varphi(t) \geq 2\sqrt{t}$.

We now study the properties of F.

Lemma 3.1. *Let $\theta \geq 0$ and $\varphi \in C_\theta([0,1],\mathbb{R})$ such that for any $t \in [0,1]$, $\varphi(t) \geq 2\sqrt{t}$ and define*

$$J(\varphi) := \sup_{h\in\mathcal{H}} F(\varphi,\sigma;h). \tag{3.4}$$

Then,

$$J(\varphi) < \infty \Rightarrow \varphi \text{ absolutely continuous}$$
$$\text{and } J(\varphi) = \frac{1}{2}\int_0^1 (\dot\varphi(s) - b(\varphi(s),\sigma_s))^2 ds = I_\theta(\varphi).$$

Proof: Recall that $F(\varphi,\sigma;h) = G(\varphi,\sigma;h) - \frac{1}{2}\int_0^1 h^2(s)ds$ where $h \mapsto G(\varphi,\sigma;h)$ is a linear functional (see (3.2)). Replacing h by λh, $\lambda \in \mathbb{R}$ and optimizing in λ yields

$$J(\varphi) = \frac{1}{2}\sup_{h\in\mathcal{H}} \frac{G^2(\varphi,\sigma;h)}{\|h\|_2^2}.$$

If $J(\varphi) < \infty$, then the linear functional $G(\varphi,\sigma,.)$ can be extended continuously to $\mathbb{L}^2([0,1])$ and by Riesz theorem, there exists $k_\varphi \in \mathbb{L}^2([0,1])$ such that $G(\varphi,\sigma,h) = \int_0^1 h(s)k_\varphi(s)ds$. Comparing with (3.2), we see that $\varphi - \int_0^\cdot b(\varphi(s),\mu_s)ds$ is absolutely continuous and $k_\varphi(s) = \dot\varphi(s) - b(\varphi(s),\sigma_s)$.

From Cauchy-Schwarz inequality, we obtain:

$$G^2(\varphi,\sigma;h) \leq ||k_\varphi||_2^2 ||h||_2^2$$

with equality if h is proportional to k_φ. Therefore, $J(\varphi) \leq \frac{1}{2}||k_\varphi||_2^2$, and the equality holds since $\mathcal{H}$ is dense in $\mathbb{L}^2([0,1])$. The equality between $\frac{1}{2}||k_\varphi||^2$ and $I_\theta(\varphi)$ follows from the computation of the Hilbert transform of the semicircular distribution recalled in (3.3) and $\varphi(t) \geq 2\sqrt{t}$. □

We can now show the following :

Proposition 3.2. *The function $I_\theta : C_\theta([0,1],\mathbb{R}) \to \mathbb{R}$ is lower semicontinuous.*

Proof: From Lemma 3.1, $I_\theta(\varphi) = \sup_{h\in\mathcal{H}} F(\varphi,\sigma;h)$ where

$$F(\varphi,\sigma;h) = h(1)\varphi(1) - h(0)\varphi(0) - \int_0^1 \varphi(s)\dot h(s)ds - \int_0^1 b(\varphi(s),\sigma_s)h(s)ds$$
$$- \frac{1}{2}\int_0^1 h^2(s)ds.$$

We shall prove that for fixed $h \in \mathcal{H}$, $\varphi \mapsto F(\varphi,\sigma;h)$ is continuous. From the definition of F, performing an integration by part in the term of the integral in b, it is enough to prove the continuity of $\varphi \mapsto \Lambda(\varphi) := \int_0^\cdot b(\varphi(s),\sigma_s)ds$ (the other terms are obviously continuous in φ). As we know that $0 \leq b(\varphi(t),\sigma_t) \leq \frac{1}{\sqrt{t}}$ for $\varphi(t) \geq 2\sqrt{t}$, by dominated convergence, if φ_n converges towards φ, $\Lambda(\varphi_n)$ converges to $\Lambda(\varphi)$ pointwise. Now, since the functions involved are increasing, the convergence holds uniformly on the compact $[0,1]$. □

4. The lower bound

In this section, we prove the large deviation lower bound, namely:

Proposition 4.1. *For any open set O in $C_\theta([0,1];\mathbb{R})$,*

$$\liminf_{N\to\infty} \frac{1}{N}\ln \mathbb{P}(\lambda_1^{\theta,N} \in O) \geq -\inf_{\varphi\in O} I_\theta(\varphi). \tag{4.1}$$

For any $\varphi \in C_\theta([0,1];\mathbb{R})$, any $\delta > 0$, $B(\varphi,\delta)$ will denote the ball centered at φ with radius δ with respect to the uniform metric, that is the subset of $C([0,1];\mathbb{R})$ of functions ψ such that $\sup_{t\in[0,1]} |\psi(t)-\varphi(t)| < \delta$. To prove Proposition 4.1, it is enough to show

Proposition 4.2.

$$\lim_{\delta\downarrow 0}\liminf_{N\to\infty} \frac{1}{N}\ln \mathbb{P}\left(\lambda_1^{\theta,N} \in B(\varphi,\delta)\right) \geq -I_\theta(\varphi) \tag{4.2}$$

for any φ belonging to a well chosen subclass $\mathcal{H}_\theta$ of $C_\theta([0,1];\mathbb{R})$ satisfying

$$\inf_{\varphi\in O\cap\mathcal{H}_\theta} I_\theta(\varphi) = \inf_{\varphi\in O} I_\theta(\varphi) \tag{4.3}$$

for any open set O.

To introduce the subclass $\mathcal{H}_\theta$, we need a few more notations.
For φ such that $I_\theta(\varphi) < \infty$, we recall from Section 3 that

$$k_\varphi(s) := \dot\varphi(s) - b(\varphi(s),\sigma_s) = \dot\varphi(s) - \frac{1}{2s}(\varphi(s) - \sqrt{\varphi^2(s)-4s}) \tag{4.4}$$

and that

$$I_\theta(\varphi) = \frac{1}{2}\int_0^1 k_\varphi^2(s)ds = \frac{1}{2}\|k_\varphi\|_2^2.$$

We define

$$\mathcal{H}_\theta = \{\varphi \in C_\theta([0,1];\mathbb{R}); \varphi(t) > 2\sqrt{t}\ \forall t\in[0,1];\ k_\varphi \text{ smooth}\} \qquad \text{for } \theta > 0,$$

$$\mathcal{H}_0 = \left\{\varphi \in C_0([0,1];\mathbb{R}); \exists t_0 > 0,\ \begin{array}{ll}\varphi(t) = 2\sqrt{t} & \text{for } \mathrm{t} \leq \mathrm{t}_0\\ \varphi(t) > 2\sqrt{t} & \text{for } \mathrm{t} > \mathrm{t}_0\end{array}\ ; k_\varphi \text{ smooth}\right\}, \tag{4.5}$$

where smooth means infinitely differentiable on $[0,1]$. For $\varphi \in \mathcal{H}_0$, we denote by $t_0(\varphi) := \sup\{t; \varphi(t) = 2\sqrt{t}\}$ the corresponding threshold.
Eq.(4.3) will be proven in Lemma 4.5 after some preliminary considerations in the next subsection. Eq.(4.2) is obtained in Section 4.4 for $\theta > 0$ and in Section 4.5 for $\theta = 0$.

4.1. *Some properties of the functions with finite entropy when $\theta = 0$.* As will be seen further, the proof that $\mathcal{H}_\theta$ is dense will be quite straightforward in the case when $\theta > 0$ but more delicate when $\theta = 0$. In this latter case, we first need to understand some features of the functions with finite entropy that we gather here.

We need the following notations : for any φ such that $\varphi(s) \geq 2\sqrt{s}$, $\forall s \in [0,1]$, we define x_φ by

$$x_\varphi(s) = \frac{\varphi(s) + \sqrt{\varphi^2(s)-4s}}{2}, \quad \forall s \in [0,1] \tag{4.6}$$

so that φ and k_φ can be reexpressed in terms of x_φ. More precisely, $\forall s \in (0,1]$,

$$\varphi(s) = x_\varphi(s) + \frac{s}{x_\varphi(s)} \tag{4.7}$$

and

$$k_\varphi(s) = 2\dot x_\varphi(s)\left(1 - \frac{s}{x_\varphi^2(s)}\right). \tag{4.8}$$

The following lemma gives the behaviour of φ near 0.

Lemma 4.3. *($\theta = 0$) Let $\varphi \in C_0([0,1])$ satisfy $I_0(\varphi) < \infty$. Then,*

$$\lim_{t\to 0} \frac{\varphi(t)}{\sqrt{t}} = 2.$$

Proof: Set

$$I^t(\varphi) = \int_0^t \left(\dot{\varphi}(s) - \frac{1}{2s}\left(\varphi(s) - \sqrt{\varphi^2(s) - 4s}\right)\right)^2 ds.$$

Then, from the finiteness of $I_0(\varphi)$, $\lim_{t\to 0} I^t(\varphi) = 0$. From Cauchy-Schwarz inequality,

$$\left|\int_0^t \dot{\varphi}(s) - \frac{1}{2s}\left(\varphi(s) - \sqrt{\varphi^2(s) - 4s}\,\right) ds\right| \leq \sqrt{t}(I^t(\varphi))^{1/2}$$

and

$$\left|\frac{\varphi(t)}{\sqrt{t}} - \frac{1}{\sqrt{t}}\int_0^t \frac{1}{2s}\left(\varphi(s) - \sqrt{\varphi^2(s) - 4s}\right) ds\right| \leq (I^t(\varphi))^{1/2}.$$

Now, we have, using (4.7),

$$0 \leq \int_0^t \frac{1}{2s}\left(\varphi(s) - \sqrt{\varphi^2(s) - 4s}\right) ds = \int_0^t \frac{ds}{x_\varphi(s)} \leq \int_0^t \frac{ds}{\sqrt{s}} = 2\sqrt{t}.$$

Thus, on one hand, $\frac{\varphi(t)}{\sqrt{t}} \geq 2$, whereas $0 \leq \frac{1}{\sqrt{t}}\int_0^t \frac{1}{2s}\left(\varphi(s) - \sqrt{\varphi^2(s) - 4s}\right) ds \leq 2$ and the difference of the two terms tends to 0 as t tends to 0. It follows that:

$$\lim_{t\to 0}\frac{\varphi(t)}{\sqrt{t}} = \lim_{t\to 0}\frac{1}{\sqrt{t}}\int_0^t \frac{1}{2s}\left(\varphi(s) - \sqrt{\varphi^2(s) - 4s}\right) ds = 2.$$

□

The following lemma will be useful in the proof of the lower bound itself.

Lemma 4.4. *($\theta = 0$) Let $\varphi \in \mathcal{H}_0$. Then k_φ is positive in a right neighborhood of $t_0(\varphi)$.*

Proof of Lemma 4.4: For $\varphi \in \mathcal{H}_0$, $k_\varphi \equiv 0$ on $[0, t_0(\varphi)]$.
Since $\varphi(s) > 2\sqrt{s}$ for $s > t_0(\varphi)$, we have that $x_\varphi(s) > \sqrt{s}$, for $s > t_0(\varphi)$. As $\dot{x}_\varphi(t_0(\varphi)) = \frac{1}{2\sqrt{t_0(\varphi)}} > 0$ and $\dot{x}_\varphi$ is continuous (as φ is smooth), $\dot{x}_\varphi(s) > 0$ in a neighborhood of $t_0(\varphi)$ and thus, from (4.8), $k_\varphi(s) > 0$ for $t_0(\varphi) < s < t_0(\varphi) + \varepsilon$ for some $\varepsilon > 0$. □

4.2. *Denseness of $\mathcal{H}_\theta$.* The goal of this subsection is to establish the following lemma

Lemma 4.5. *Let $\varphi \in C_\theta([0,1])$ satisfying $I_\theta(\varphi) < \infty$. There exists a sequence $(\varphi_p)_{p\in\mathbb{N}^*}$ of functions in $\mathcal{H}_\theta$ such that, as p goes to infinity,*
- *φ_p converges to φ in $C_\theta([0,1],\mathbb{R})$*
- *$I_\theta(\varphi_p)$ converges to $I_\theta(\varphi)$. Therefore, (4.3) holds.*

4.2.1. *Proof of Lemma 4.5 when $\theta > 0$.* Let φ such that $I_\theta(\varphi) < \infty$. As $\varphi(0) = \theta > 0$ and φ is continuous, there exists $t_1 > 0$ such that for any $t \in [0, t_1]$, $\varphi(t) > 2\sqrt{t}$.

For any $p \in \mathbb{N}^*$, we define

$$\chi_p(t) = \begin{cases} \varphi(t) & \text{if } t \leq t_1, \\ \varphi(t) + (t - t_1) & \text{if } t_1 \leq t \leq t_1 + \frac{1}{p}, \\ \varphi(t) + \frac{1}{p} & \text{if } t \geq t_1 + \frac{1}{p}. \end{cases}$$

It is easy to check that χ_p is continuous and for p large enough, for any $t \in [0, 1]$, $\chi_p(t) > 2\sqrt{t}$ and $I_\theta(\chi_p) < \infty$. Moreover, as p goes to infinity, χ_p converges to φ in the uniform norm and k_{χ_p} converges to k_φ in $\mathbb{L}^2([0, 1])$.

It is now enough to check that χ_p can be approximated by a sequence of functions in $\mathcal{H}_\theta$. As we know that for any $t \in [0, 1]$, $\chi_p(t) > 2\sqrt{t}$, we have that $\inf_{s \in (0,1]} \left(1 - \frac{s}{x^2_{\chi_p}(s)}\right) > 0$. As $k_{\chi_p} \in \mathbb{L}^2([0, 1])$, from (4.8) we get that so does $\dot{x}_{\chi_p}$. It can be approximated by a sequence of smooth functions $\dot{x}_{p,q}$. Set $x_{p,q}(t) = \theta + \int_0^t \dot{x}_{p,q}(s)ds$. The corresponding $\chi_{p,q}$ is defined by

$$\chi_{p,q}(s) = x_{p,q}(s) + \frac{s}{x_{p,q}(s)}$$

and

$$k_{p,q}(s) = 2\dot{x}_{p,q}(s) \left(1 - \frac{s}{x^2_{p,q}(s)}\right),$$

so that $k_{p,q}$ is smooth. For q large enough, for any $s \in [0, 1]$, $x_{p,q}(s) > \sqrt{s}$, so that $\chi_{p,q}(s) > 2\sqrt{s}$.

Moreover, as q grows to infinity, the sequence $x_{p,q}$ converges towards x_{χ_p} in uniform norm on $[0, 1]$ so that $\chi_{p,q}$ converges towards χ_p in the same sense and $k_{p,q}$ converges to k_{χ_p} in $\mathbb{L}^2([0, 1])$.

To conclude the proof of the lemma, it is enough to notice that one can find an increasing function ψ from $\mathbb{N}$ to $\mathbb{N}$ such that $\varphi_p := \chi_{p,\psi(p)} \in \mathcal{H}_\theta$ converges towards φ and $k_{\varphi_p} = k_{p,\psi(p)}$ converges to k_φ in $\mathbb{L}^2([0, 1])$.

4.2.2. *Proof of Lemma 4.5 when $\theta = 0$.* As in the latter paragraph, we establish the proof in two steps: first, we approximate φ by a sequence of functions that are equal to $2\sqrt{t}$ in a neighborhood of 0 and strictly greater than $2\sqrt{t}$ away from 0. Next, we approximate those functions by smooth ones.

Let $r > 0$ and define χ_r by:

$$\chi_r(t) = \begin{cases} 2\sqrt{t} & t \leq y_r^2, \\ y_r + \dfrac{t}{y_r} & y_r^2 \leq t \leq r, \\ \varphi(t) + r(t - r), & t \geq r \end{cases}$$

with $y_r = \frac{\varphi(r) - \sqrt{\varphi^2(r) - 4r}}{2} \leq \sqrt{r}$ so that χ_r is continuous.

$$\|\varphi - \chi_r\| \leq \sup_{s \leq r} |\varphi(s) - \chi_r(s)| \vee r \leq 2 \sup_{s \leq r} |\varphi(s) - 2\sqrt{s}| \vee r \to_{r \to 0} 0$$

using Lemma 4.3. It remains to show that $I_0(\varphi) - I_0(\chi_r)$ tends to 0. If we set $J_r(f) = \int_0^r k_f^2(s)ds$ and $J'_r(f) = \int_r^1 k_f^2(s)ds$, we get

$$I_0(\varphi) - I_0(\chi_r) = (J_r(\varphi) - J_r(\chi_r)) + (J'_r(\varphi) - J'_r(\chi_r))$$

with $J_r(\varphi) \to 0$ as $r \to 0$.

$$J_r(\chi_r) = \int_{y_r^2}^r \left(\frac{1}{y_r} - \frac{y_r}{s}\right)^2 ds = \frac{r}{y_r^2} - \frac{y_r^2}{r} - 2\ln\left(\frac{r}{y_r^2}\right) \to_{r\to 0} 0$$

since $y_r/\sqrt{r}$ tends to 1 thanks to Lemma 4.3.

On the other hand, if we define $h_{\varphi,r}$ by

$$h_{\varphi,r}(t) := r - \frac{1}{2t}\left(r(t-r) + \sqrt{\varphi^2(t) - 4t} - \sqrt{(\varphi(t) + r(t-r))^2 - 4t}\right),$$

then, by Cauchy-Schwarz inequality, we have

$$|J_r'(\varphi) - J_r'(\chi_r)|^2 \le \int_r^1 (2k_\varphi(s) + h_{\varphi,r}(t))^2 dt \int_r^1 (h_{\varphi,r}(t))^2 dt.$$

Therefore it is enough to show that $\int_r^1 (h_{\varphi,r}(t))^2 dt$ goes to zero as r goes to zero.

To show that, we notice that, for $t \in [r,1]$,

$$\left|\frac{r(t-r)}{2t}\right| \le \frac{r}{2}. \tag{4.9}$$

Moreover,

$$\begin{aligned}
\left|\frac{1}{2t}\left(\sqrt{\varphi^2(t) - 4t}\right.\right. &- \left.\left.\sqrt{(\varphi(t) + r(t-r))^2 - 4t}\right)\right| \qquad (4.10)\\
&= \frac{1}{2t}\frac{2r\varphi(t)(t-r) + r^2(t-r)^2}{\sqrt{\varphi^2(t) - 4t} + \sqrt{(\varphi(t) + r(t-r))^2 - 4t}}\\
&\le \frac{\sqrt{2r\varphi(t)(t-r)}}{2t} + \frac{|r(t-r)|}{2t}\\
&\le Cr^{1/4} + \frac{r}{2}, \qquad (4.11)
\end{aligned}$$

where we used that, from Lemma 4.3, $t \mapsto \frac{\varphi(t)}{\sqrt{t}}$ is bounded on $[0,1]$ by a constant C. Putting (4.9) and (4.11) together, we get $\int_r^1 (h_{\varphi,r}(t))^2 dt$ goes to zero as r goes to zero.

Now $\dot\chi_r(s) = \frac{1}{\sqrt{s}}$ on $[0, y_r^2]$ and $\dot\chi_r \in \mathbb{L}^2([y_r^2, 1])$ since $k_{\chi_r} \in \mathbb{L}^2([y_r^2, 1])$. For any $r > 0$, there exists a sequence of function $\dot\chi_{r,q}$ smooth on $]0,1]$ such that $\dot\chi_{r,q}(s) = \frac{1}{\sqrt{s}}$ on $[0, y_r^2]$ and $\dot\chi_{r,q}$ tends to $\dot\chi_r$ in $\mathbb{L}^2([y_r^2/2, 1])$. Setting $\chi_{r,q}(t) = \int_0^t \dot\chi_{r,q}(s)ds$, we have:

- $\chi_{r,q}$ tends to χ_r in uniform norm.
- $k_{\chi_{r,q}}$ is smooth.
- $k_{\chi_{r,q}}$ converges to k_{χ_r} in $\mathbb{L}^2([0,1])$ so that $I_0(\chi_{r,q})$ converges to $I_0(\chi_r)$.

Putting everything together, we conclude that there exists an increasing function ψ such that the sequence of functions $\varphi_p = \chi_{p,\psi(p)}$ satisfies the requirements of Lemma 4.5. □

4.3. *Almost sure convergence of* $\lambda_1^{\theta,N}$ *and* ν_N *under* $\mathbb{P}^{k_\varphi}$. The strategy of the proof of the lower bound will be classical : we will make a proper change of measure so that the function φ becomes "typical" under the new measure. We will therefore need to study more precisely the behavior of $\lambda_1^{\theta,N}$ under the new measure $\mathbb{P}^{k_\varphi}$.

To be more precise, for $h \in \mathbb{L}^2([0,1])$, we define the exponential martingale M^h such that for any $t \in [0,1]$,

$$M_t^h = \exp\left[N\left(\int_0^t h(s)\frac{1}{\sqrt{N}}dB_1(s) - \frac{1}{2}\int_0^t h^2(s)ds\right)\right], \tag{4.12}$$

where B_1 is the standard Brownian motion appearing in the SDE for $\lambda_1^{\theta,N}$ (see (1.1)). We denote by $(\mathcal{F}_t)_{t\geq 0}$ its canonical filtration.

We now introduce $\mathbb{P}^{k_\varphi}$ the probability defined by $\mathbb{P}^{k_\varphi} := M_1^{k_\varphi}\sharp\mathbb{P}$, meaning that for any $t \leq 1$, the Radon-Nikodym derivative of $\mathbb{P}^{k_\varphi}$ with respect to $\mathbb{P}$ on $\mathcal{F}_t$ is given by $M_t^{k_\varphi}$ and we also denote by $\mathbb{E}^{k_\varphi}$ the expectation under $\mathbb{P}^{k_\varphi}$. Recall that ν_N is the empirical distribution of all but the largest eigenvalues defined in the introduction. For any $r > 0, \alpha > 0$, we also define

$$\mathbb{B}_r(\sigma,\alpha) := \mathbb{B}(\sigma,\alpha)\bigcap\{\mu \in C([0,1],\mathcal{P}(\mathbb{R}));\ \forall s,\ \mathrm{supp}(\mu_s) \subset]-\infty, 2\sqrt{s}+r]\}.$$

The goal of this subsection will be to show

Proposition 4.6. *For any $r > 0, \delta > 0, \alpha > 0$ and $\varphi \in \mathcal{H}_\theta$,*

$$\mathbb{P}^{k_\varphi}(\lambda_1^{\theta,N} \in B(\varphi,\delta); \nu_N \in \mathbb{B}_r(\sigma,\alpha))) \xrightarrow[N\to\infty]{} 1.$$

The proof of the proposition relies on some lemmata.

Lemma 4.7. *Under $\mathbb{P}^{k_\varphi}$, μ_N and ν_N converge as N goes to infinity to the semi-circular process σ.*

Proof: It is well known that μ_N is exponentially tight in scale N^2, under $\mathbb{P}$ (see Cabanal Duvillard and Guionnet (2001), Guionnet (2009, Chap. 12)). Let $A \in C([0,1],\mathcal{P}(\mathbb{R}))$, then

$$\begin{aligned}\mathbb{P}^{k_\varphi}(\mu_N \in A) &= \mathbb{E}\left(M_1^{k_\varphi}\mathbf{1}_{\mu_N\in A}\right)\\ &\leq \mathbb{E}((M_1^{k_\varphi})^2)^{1/2}(\mathbb{P}(\mu_N \in A))^{1/2}\\ &= \exp\left(\frac{N}{2}\int_0^1 k_\varphi^2(s)ds\right)\mathbb{P}(\mu_N \in A)^{1/2}.\end{aligned} \tag{4.13}$$

From the exponential tighness of μ_N under $\mathbb{P}$, there exists a compact K_L in $C([0,1],\mathcal{P}(\mathbb{R}))$ such that

$$\mathbb{P}(\mu_N \in K_L^c) \leq \exp(-N^2 L).$$

Therefore, from (4.13)

$$\mathbb{P}^{k_\varphi}(\mu_N \in K_L^c) \leq \exp(-N^2L/4)$$

for N large enough. This proves the exponential tightness of μ_N under $\mathbb{P}^{k_\varphi}$ and thus its a.s. pre-compactness in $C([0,1],\mathcal{P}(\mathbb{R}))$. It remains to prove the uniqueness of any limit point.

From Girsanov's theorem, we have that under $\mathbb{P}^{k_\varphi}$, the process $(\lambda_i^{\theta,N}(t))_{t\leq 1, i=1,\ldots N}$ satisfies the system of stochastic differential equations:

$$\begin{cases} d\lambda_i(t) = \dfrac{1}{\sqrt{N}}d\beta_i(t) + \dfrac{1}{N}\displaystyle\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}dt, & i = 2,\ldots,N\\ d\lambda_1(t) = \dfrac{1}{\sqrt{N}}d\beta_1(t) + k_\varphi(t)dt + \dfrac{1}{N}\displaystyle\sum_{j\neq 1}\frac{1}{\lambda_1(t)-\lambda_j(t)}dt \end{cases} \tag{4.14}$$

where $(\beta_i)_{1\leq i\leq N}$ are independent Brownian motions under $\mathbb{P}^{k_\varphi}$. The proof of the uniqueness of any limit point follows the same proof as in Rogers and Shi (1993, Theorem 1) (see also Guionnet (2009, Chap. 12), Chan (1992)): let $f \in C_b^2(\mathbb{R})$, using Itô's formula and (4.14), we obtain a stochastic differential equation for $\langle \mu_N(t), f\rangle$ with a diffusion coefficient tending to 0 as n tends to ∞. Then, any limit point μ_t satisfies a deterministic evolution equation (the term in k_φ disappears in the limit)

$$\langle \mu_t, f\rangle = \int f(x)d\mu_t(x) = \int f(x)d\mu_0(x) + \frac{1}{2}\int_0^t \int \frac{f'(x)-f'(y)}{x-y} d\mu_s(x)d\mu_s(y)ds$$

for which uniqueness holds. When $\mu_0 = \delta_0$ as in our setting, μ_t is the semicircular law σ_t. Therefore μ_N converges a.s. to the semicircle process σ. Since $d(\mu_N, \nu_N) \leq \frac{2}{N}$, the same convergence holds for ν_N. □

Lemma 4.8. *For any $r > 0, \alpha > 0$,*

$$\mathbb{P}^{k_\varphi}(\nu_N \in \mathbb{B}_r(\sigma, \alpha)) \xrightarrow[N\to\infty]{} 1.$$

Proof: Since we already know the convergence of ν_N towards σ under $\mathbb{P}^{k_\varphi}$, it is enough to prove that under $\mathbb{P}^{k_\varphi}$,

$$\limsup_{N\to\infty} \lambda_2(t) \leq 2\sqrt{t}. \tag{4.15}$$

We define $(\lambda_i^{(\varepsilon)}(t), i = 1, \ldots N)$ the strong solution of the system of SDE (4.14) with initial conditions $\lambda_1^{(\varepsilon)}(0) = \theta + \varepsilon$ and $\lambda_i^{(\varepsilon)}(0) = \frac{\varepsilon}{i}$, for $i = 2, \ldots, N$, so that in particular $\lambda_2(t) = \lambda_2^{(0)}(t)$.

We also introduce $(\overline{\lambda_i^{(\varepsilon)}}, i = 2, \ldots, N)$ the strong solution of the system of SDE:

$$d\overline{\lambda_i^{(\varepsilon)}}(t) = \frac{1}{\sqrt{N}} d\beta_i(t) + \frac{1}{N}\sum_{j=2, j\neq i}^{N} \frac{1}{\overline{\lambda_i^{(\varepsilon)}}(t) - \overline{\lambda_j^{(\varepsilon)}}(t)} dt, \qquad i = 2, \ldots, N, \tag{4.16}$$

with initial conditions $\overline{\lambda_i^{(\varepsilon)}}(0) = \frac{\varepsilon}{i}$, for $i = 2, \ldots, N$.

The process $(\overline{\lambda_i^{(0)}}, i = 2, \ldots, N)$ is distributed as the eigenvalues of $\sqrt{\frac{N-1}{N}} H_{N-1}(t)$ where H_{N-1} is a standard Hermitian Brownian motion of size $N-1$. Therefore, $\lim_{N\to\infty} \overline{\lambda_2^{(0)}}(t) = 2\sqrt{t}$ a.s.

Our goal is now to compare $\lambda_2^{(0)}(t)$ with $\overline{\lambda_2^{(0)}}(t)$.

The first step is to show that for any $\varepsilon > 0$ fixed, N fixed, for all $t \in [0, 1]$, $\lambda_2^{(\varepsilon)}(t) \leq \overline{\lambda_2^{(\varepsilon)}}(t)$. In fact, we will show that for any $2 \leq i \leq N$, we have $\lambda_i^{(\varepsilon)}(t) \leq \overline{\lambda_i^{(\varepsilon)}}(t)$, for all $t \in [0, 1]$.

Let $R > 0$ large enough so that $\frac{1}{R} < \frac{\varepsilon}{N^2}$ and

$$\begin{aligned} T_R \quad = \quad \inf \Big\{ & t \geq 0, \forall i, j = 2, \ldots, N, i \neq j, \\ & \left|\lambda_i^{(\varepsilon)}(t) - \lambda_j^{(\varepsilon)}(t)\right| \vee \left|\overline{\lambda_i^{(\varepsilon)}}(t) - \overline{\lambda_j^{(\varepsilon)}}(t)\right| \leq \frac{1}{R} \Big\}. \end{aligned}$$

For any $t \geq 0$ and $2 \leq i \leq N$, we can write

$$d\lambda_i^{(\varepsilon)}(t) = \frac{1}{\sqrt{N}} d\beta_i(t) + f_i(\boldsymbol{\lambda}^{(\varepsilon)}(t))dt,$$

and

$$d\overline{\lambda_i^{(\varepsilon)}}(t) = \frac{1}{\sqrt{N}} d\beta_i(t) + g_i(\overline{\boldsymbol{\lambda}^{(\varepsilon)}}(t))dt,$$

with $\boldsymbol{\lambda}^{(\varepsilon)}(t) := (\lambda_1^{(\varepsilon)}(t), \ldots, \lambda_N^{(\varepsilon)}(t))$, $\overline{\boldsymbol{\lambda}^{(\varepsilon)}}(t) := (\overline{\lambda_2^{(\varepsilon)}}(t), \ldots, \overline{\lambda_N^{(\varepsilon)}}(t))$ and $f_i : \mathbb{R}^N \to \mathbb{R}$ and $g_i : \mathbb{R}^{N-1} \to \mathbb{R}$ are such that

$$f_i(x_1, \ldots, x_N) = \frac{1}{N} \frac{1}{x_i - x_1} + g_i(x_2, \ldots, x_N).$$

We also denote by $\widetilde{\boldsymbol{\lambda}^{(\varepsilon)}}(t) := (\lambda_2^{(\varepsilon)}(t), \ldots, \lambda_N^{(\varepsilon)}(t))$. Then

$$d\left(\lambda_i^{(\varepsilon)} - \overline{\lambda_i^{(\varepsilon)}}\right)(t) = (f_i(\boldsymbol{\lambda}^{(\varepsilon)}(t)) - g_i(\overline{\boldsymbol{\lambda}^{(\varepsilon)}}(t))dt \leq (g_i(\widetilde{\boldsymbol{\lambda}^{(\varepsilon)}}(t)) - g_i(\overline{\boldsymbol{\lambda}^{(\varepsilon)}}(t)))dt,$$

as $\lambda_i^{(\varepsilon)}(t) - \lambda_1^{(\varepsilon)}(t) \leq 0$.

We denote by $x^+ = \max(0, x)$ so that

$$d\left(\lambda_i^{(\varepsilon)} - \overline{\lambda_i^{(\varepsilon)}}\right)^+(t) \leq \mathbf{1}_{\overline{\lambda_i^{(\varepsilon)}}(t) \leq \lambda_i^{(\varepsilon)}(t)} (g_i(\widetilde{\boldsymbol{\lambda}^{(\varepsilon)}}(t)) - g_i(\overline{\boldsymbol{\lambda}^{(\varepsilon)}}(t)))dt.$$

Now

$$g_i(\widetilde{\boldsymbol{\lambda}^{(\varepsilon)}}(t)) - g_i(\overline{\boldsymbol{\lambda}^{(\varepsilon)}}(t)) = \frac{1}{N} \sum_{\substack{k \neq i \\ k \geq 2}} \frac{(\overline{\lambda_i^{(\varepsilon)}} - \lambda_i^{(\varepsilon)}) - (\overline{\lambda_k^{(\varepsilon)}} - \lambda_k^{(\varepsilon)})}{(\lambda_i^{(\varepsilon)} - \lambda_k^{(\varepsilon)})(\overline{\lambda_i^{(\varepsilon)}} - \overline{\lambda_k^{(\varepsilon)}})}(t)$$

As the eigenvalues are ordered, the denominator is always positive so that, for all $i \geq 2$,

$$\mathbf{1}_{\overline{\lambda_i^{(\varepsilon)}}(t) \leq \lambda_i^{(\varepsilon)}(t)} \frac{(\overline{\lambda_i^{(\varepsilon)}} - \lambda_i^{(\varepsilon)})}{(\lambda_i^{(\varepsilon)} - \lambda_k^{(\varepsilon)})(\overline{\lambda_i^{(\varepsilon)}} - \overline{\lambda_k^{(\varepsilon)}})}(t) \leq 0.$$

On the other hand, for all $k \neq i$, for $t \in [0, T_R]$,

$$\frac{-(\overline{\lambda_k^{(\varepsilon)}} - \lambda_k^{(\varepsilon)})}{(\lambda_i^{(\varepsilon)} - \lambda_k^{(\varepsilon)})(\overline{\lambda_i^{(\varepsilon)}} - \overline{\lambda_k^{(\varepsilon)}})}(t) \leq R^2(\lambda_k^{(\varepsilon)} - \overline{\lambda_k^{(\varepsilon)}})^+(t),$$

so that on $[0, T_R]$,

$$d\left(\lambda_i^{(\varepsilon)} - \overline{\lambda_i^{(\varepsilon)}}\right)^+(t) \leq \frac{R^2}{N} \sum_{k \neq i} (\lambda_k^{(\varepsilon)} - \overline{\lambda_k^{(\varepsilon)}})^+(t)dt,$$

and if we sum over the index i,

$$d\left(\sum_{i=2}^{N} (\lambda_i^{(\varepsilon)} - \overline{\lambda_i^{(\varepsilon)}})\right)^+(t) \leq \frac{R^2(N-1)}{N} \sum_{k=2}^{N} (\lambda_k^{(\varepsilon)} - \overline{\lambda_k^{(\varepsilon)}})^+(t)dt.$$

By Gronwall lemma, we get that on $[0, T_R]$,

$$\sum_{i=2}^{N} (\lambda_i^{(\varepsilon)} - \overline{\lambda_i^{(\varepsilon)}})(t) = 0,$$

which means that for all $i \geq 2$, $\lambda_i^{(\varepsilon)}(t) \leq \overline{\lambda_i^{(\varepsilon)}}(t)$.

Moreover, from Cépa and Lépingle (1997), we know that T_R goes to infinity as R goes to infinity. In particular, if we choose R large enough for T_R to be larger than 1, our inequalities hold for any $t \in [0, 1]$.

Now, the solutions of (4.16), resp. (4.14), are continuous with respect to the initial condition (see Cépa (1994)); thus, letting $\varepsilon \to 0$, we obtain $\lambda_2^{(0)}(t) \leq \overline{\lambda_2^{(0)}}(t)$ a.s. Putting everything together, we have that $\limsup_{N\to\infty} \lambda_2(t) \leq 2\sqrt{t}$. $\square$

Lemma 4.9. *Let $\varphi \in \mathcal{H}_\theta$.*

1) *For $\theta > 0$, the differential equation $dy(t) = (k_\varphi(t) + b(y(t), \sigma_t))dt$ on $[0,1]$ with initial value $y(0) = \theta$ admits a unique solution larger than $2\sqrt{t}$, namely φ.*
2) *For $\theta = 0$, the differential equation $dy(t) = (k_\varphi(t) + b(y(t), \sigma_t))dt$ on $[0,1]$ with value $y(t_0(\varphi)) = 2\sqrt{t_0(\varphi)}$ at time $t_0(\varphi)$ admits a unique solution larger than $2\sqrt{t}$, namely φ.*

Proof: Let us first check that in both cases there is a unique solution larger than $2\sqrt{t}$. We recall that for $x \geq 2\sqrt{t}$,

$$b(x, \sigma_t) = \frac{1}{2\pi t} \int \frac{1}{x-y}\sqrt{4t - y^2}dy = \frac{1}{2t}(x - \sqrt{x^2 - 4t}).$$

It is easy to see that $x \mapsto b(x, \sigma_t)$ is decreasing on $[2\sqrt{t}, \infty[$. Let x, y two solutions of $dy(t) = (k_\varphi(t) + b(y(t), \sigma_t))dt$ such that for any $t \in [0,1]$, $x(t), y(t) \geq 2\sqrt{t}$. Then,

$$(x(t) - y(t))^2 = 2\int_0^t (x(s) - y(s))(b(x(s), \sigma_s) - b(y(s), \sigma_s))ds \leq 0.$$

In the first case, it is very easy to check that φ is a solution. In the second case, notice that for $t \leq t_0(\varphi)$, $k_\varphi(t) = 0$ and we know that $t \mapsto 2\sqrt{t}$ is a solution of $dy(t) = b(y(t), \sigma_t)dt$ with initial condition $y(0) = 0$. $\square$

The last lemma to complete the proof of Proposition 4.6 is the following

Lemma 4.10. *For any $\theta \geq 0$ and $\varphi \in \mathcal{H}_\theta$, under $\mathbb{P}^{k_\varphi}$, the process $\lambda_1^{\theta,N}$ converges a.s. to φ.*

Proof : As in Lemma 4.7, from the exponential tightness in scale N of λ_1 under $\mathbb{P}$, we deduce the exponential tighness of λ_1 under $\mathbb{P}^{k_\varphi}$ and the a.s. pre-compactness of λ_1. Let $x(t)$ be a limit point. There exists $f : \mathbb{N} \to \mathbb{N}$ strictly increasing such that $\lambda_1^{\theta, f(N)}(t)$ converge to $x(t)$. In the sequel we omit the superscript $\theta, f(N)$.

The crucial step of the proof, which is similar for any value of θ is to show that $x(t) \geq \varphi(t)$.

From the a.s. convergence of $(\mu_N)_t$ towards σ_t and using that $\sigma_t([2\sqrt{t} - \varepsilon, 2\sqrt{t}]) > 0$, it follows that $\liminf_N \lambda_1(t) \geq 2\sqrt{t}$ and thus $x(t) \geq 2\sqrt{t}$. From Itô's formula, we get

$$\begin{aligned}
((\varphi(t) - \lambda_1(t))^+)^2 \quad &= \quad -\frac{2}{\sqrt{N}}(\varphi(t) - \lambda_1(t))^+ d\beta_1(t) \\
&\quad +2\int_0^1 (\varphi(t) - \lambda_1(t))^+[b(\varphi(t), \sigma_t) - b_N(\lambda_1(t), (\nu_N)_t)]dt \\
&\quad +\frac{1}{N}\mathbf{1}_{\varphi(t) - \lambda_1(t) \geq 0}dt
\end{aligned}$$

where $b_N = \frac{N-1}{N}b$.

The first and last term converge to zero and we decompose the second term in three $2(A^1(t) + A^2(t) + A^3(t))$ where

$$A^1(t) = \int_0^1 (\varphi(t) - \lambda_1(t))^+[b(\varphi(t), \sigma_t) - b_N(\varphi(t), \sigma_t)]dt$$
$$= \frac{1}{N}\int_0^1 (\varphi(t) - \lambda_1(t))^+ b(\varphi(t), \sigma_t)dt,$$

$$A^2(t) = \frac{N-1}{N}\int_0^1 (\varphi(t) - \lambda_1(t))^+[b(\varphi(t), \sigma_t) - b(\lambda_1(t), \sigma_t)]dt$$

and

$$A^3(t) = \frac{N-1}{N}\int_0^1 (\varphi(t) - \lambda_1(t))^+[b(\lambda_1(t), \sigma_t) - b(\lambda_1(t), (\nu_N)_t)]dt.$$

Passing to the limit, we obtain:

$$((\varphi(t) - x(t))^+)^2 = 2 \lim_{N\to\infty}(A^1(t) + A^2(t) + A^3(t)) \text{ a.s..}$$

We now use the continuity on $\mathbb{R}$ and the monotony on $[2\sqrt{t}, \infty[$ of the function $x \mapsto b(x, \sigma_t)$, the lower semicontinuity of $(x, \mu) \mapsto b(x, \mu)$ to conclude that :

$$\lim_{N\to\infty} A^2(t) = \int_0^1 (\varphi(t) - x(t))^+[b(\varphi(t), \sigma_t) - b(x(t), \sigma_t)]dt \le 0 \text{ a.s.}$$

and

$$\lim_{N\to\infty} A^3(t) \le 0 \text{ a.s..}$$

We also easily get that

$$\lim_{N\to\infty} A^1(t) = 0 \text{ a.s.}$$

since $b(\varphi(t), \sigma_t) \in L^1([0,1])$ for $\varphi \in \mathcal{H}_\theta$. Therefore, we obtain that $x(t) \ge \varphi(t)$.

In the case when $\theta > 0$, we therefore get that $x(t)$ is well separated from the support of σ_t, we can argue as before, using (4.15), and obtain that $\lim_{N\to\infty} b_N(\lambda_1(t), (\nu_N)_t) = b(x(t), \sigma_t)$. Therefore, letting $N \to \infty$ in the equation of λ_1, we obtain that x is a solution of the differential equation $dy(t) = (k_\varphi(t) + b(y(t), \sigma_t))dt$ and therefore equal to φ.

In the case when $\theta = 0$, we have to treat first the case $t \le t_0$. On this interval, $k_\varphi = 0$, so that $\lambda_1(t)$ converges to $2\sqrt{t}$, that is $\varphi(t)$.

For any $t > t_0$, $x(t)$ is well separated from the support of σ_t, and we get as before $\lim_{N\to\infty} b_N(\lambda_1(t), (\nu_N)_t) = b(x(t), \sigma_t)$ so that x is a solution of the differential equation $dy(t) = (k_\varphi(t) + b(y(t), \sigma_t))dt$ with initial condition $x(t_0) = 2\sqrt{t_0}$ and therefore equal to φ. □

Proposition 4.6 is straightforward from Lemmata 4.7 to 4.10.

4.4. *Lower bound for a non null initial condition: $\theta > 0$.* We want to show Proposition 4.2 - Eq.(4.2) for $\varphi \in \mathcal{H}_\theta$ under the assumption that $\theta > 0$.

We set $r := \frac{1}{2}\inf_{s\in[0,1]}(\varphi(s) - 2\sqrt{s}) > 0$.

From our assumptions on φ, there exists $\delta > 0$ small enough such that:

$$\forall \chi \in B(\varphi, \delta), \forall \mu \in \mathbb{B}_r(\sigma, \alpha), \forall s \in]0,1] \text{ and } y \in \text{supp}(\mu_s), \chi(s) - y \ge \frac{r}{4}. \quad (4.17)$$

For $h \in \mathcal{H}$ and $(\varphi, \mu) \in C_\theta([0,1];\mathbb{R}) \times C([0,1];\mathcal{P}(\mathbb{R}))$ such that for all $t \in [0,1]$, $\varphi(t) > r(\mu_t)$, we can define

$$\begin{aligned} G_N(\varphi,\mu;h) &= h(1)\varphi(1) - h(0)\varphi(0) - \int_0^1 \varphi(s)\dot{h}(s)ds - \int_0^1 b_N(\varphi(s),\mu_s)h(s)ds, \\ F_N(\varphi,\mu;h) &:= G_N(\varphi,\mu;h) - \frac{1}{2}\int_0^1 h^2(s)ds \end{aligned} \tag{4.18}$$

where we recall that $b_N = \dfrac{N-1}{N} b$.
Therefore, from (4.12) and (1.1), we have

$$M_1^h = \exp(NF_N(\lambda_1, \nu_N; h)). \tag{4.19}$$

We get

$$\begin{aligned} \mathbb{P}(\lambda_1 \in B(\varphi,\delta)) &\geq \mathbb{P}\left(\lambda_1 \in B(\varphi,\delta); \nu_N \in \mathbb{B}_r(\sigma,\alpha)\right) \\ &= \mathbb{E}\left(\mathbf{1}_{\lambda_1 \in B(\varphi,\delta);\nu_N \in \mathbb{B}_r(\sigma,\alpha)} \frac{M_1^{k_\varphi}}{M_1^{k_\varphi}}\right) \\ &= \mathbb{E}^{k_\varphi}\left(\mathbf{1}_{\lambda_1 \in B(\varphi,\delta);\nu_N \in \mathbb{B}_r(\sigma,\alpha)} \exp(-NF_N(\lambda_1,\nu_N;k_\varphi))\right) \\ &\geq \exp\left(-N \sup_{(\psi,\mu)\in C_{\alpha,\delta,r}} F_N(\psi,\mu;k_\varphi)\right) \\ &\quad \times \mathbb{P}^{k_\varphi}\left(\lambda_1 \in B(\varphi,\delta); \nu_N \in \mathbb{B}_r(\sigma,\alpha)\right) \end{aligned}$$

where

$$C_{\alpha,\delta,r} = B(\varphi,\delta) \times \mathbb{B}_r(\sigma,\alpha). \tag{4.20}$$

Therefore

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi,\delta)) &\geq - \sup_{(\psi,\mu)\in C_{\alpha,\delta,r}} F(\psi,\mu;k_\varphi) \\ &+ \liminf_{N\to\infty} \frac{1}{N} \ln \mathbb{P}^{k_\varphi}(\lambda_1 \in B(\varphi,\delta); \nu_N \in \mathbb{B}_r(\sigma,\alpha)). \end{aligned} \tag{4.21}$$

From the property (4.17) above, the fonction $(\psi,\mu) \mapsto F(\psi,\mu;k_\varphi)$ is continuous on $C_{\alpha,\delta,r}$ and we checked in Lemma 3.1 that $F(\varphi,\sigma;k_\varphi) = I_\theta(\varphi)$.

Moreover, from Proposition 4.6, we get that the last term in (4.21) is equal to zero.

We have thus obtained that for $\varphi \in \mathcal{H}_\theta$,

$$\lim_{\delta\downarrow 0} \liminf_{N\to\infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1^{\theta,N} \in B(\varphi,\delta)) \geq -I_\theta(\varphi). \tag{4.22}$$

4.5. *Lower bound for a null initial condition.* We want to show Proposition 4.2 - Eq.(4.2) under the assumption that $\theta = 0$.

Let $\varphi \in \mathcal{H}_0$ and we set $t_0(\varphi)$ as defined in (4.5). Then, $k_\varphi = 0$ on $[0, t_0(\varphi)]$. We choose ε given by Lemma 4.4 and we denote by $r := r(\varepsilon) = \frac{1}{2}\inf_{s\in[t_0(\varphi)+\varepsilon,1]}(\varphi(s) - 2\sqrt{s}) > 0$. As in the case when $\theta > 0$, we perform a change of measure via the martingale M^{k_φ}. Recall that F_N is defined by (4.18). We define $F_N^{(\varepsilon)}$ by

$$F_N^{(\varepsilon)}(\varphi,\mu;k_\varphi) = k_\varphi(1)\varphi(1) - \int_{t_0(\varphi)}^1 \varphi(s)\dot{k}_\varphi(s)ds - \int_{t_0(\varphi)+\varepsilon}^1 b_N(\varphi(s),\mu_s)k_\varphi(s)ds - \frac{1}{2}\int_0^1 k_\varphi^2(s)ds,$$

in other words,

$$F_N(\varphi,\mu;k_\varphi) = F_N^{(\varepsilon)}(\varphi,\mu;k_\varphi) - \int_{t_0(\varphi)}^{t_0(\varphi)+\varepsilon} b_N(\varphi(s),\mu_s)k_\varphi(s)ds.$$

Therefore, for such ε, $F_N \le F_N^{(\varepsilon)}$ and we obtain (as in the previous subsection)

$$\mathbb{P}(\lambda_1 \in B(\varphi,\delta)) \ge \exp\left(-N \sup_{(\psi,\mu)\in C_{\alpha,\delta,r}} F_N^{(\varepsilon)}(\psi,\mu;k_\varphi)\right) \mathbb{P}^{k_\varphi}(\lambda_1 \in B(\varphi,\delta); \nu_N \in \mathbb{B}_r(\sigma,\alpha))$$

where $C_{\alpha,\delta,r}$ is defined in (4.20) and, using Proposition 4.6,

$$\liminf_{N\to\infty} \frac{1}{N}\ln \mathbb{P}(\lambda_1 \in B(\varphi,\delta)) \ge - \sup_{(\psi,\mu)\in C_{\alpha,\delta,r}} F_N^{(\varepsilon)}(\psi,\mu;k_\varphi).$$

Now, for $\delta = \delta(\varepsilon)$ small enough, $F_N^{(\varepsilon)}$ is continuous on $C_{\alpha,\delta,r}$, since

$$\forall \psi \in B(\varphi,\delta), \forall \mu \in \mathbb{B}_r(\sigma,\alpha), \forall s \in [t_0(\varphi)+\varepsilon, 1] \text{ and } y \in \operatorname{supp}(\mu_s), \psi(s) - y \ge \frac{r}{4}$$

and therefore

$$\lim_{\delta\to 0}\liminf_{N\to\infty} \frac{1}{N}\ln \mathbb{P}(\lambda_1 \in B(\varphi,\delta)) \ge -F^{(\varepsilon)}(\varphi,\sigma;k_\varphi),$$

where

$$F^{(\varepsilon)}(\varphi,\sigma;k_\varphi) = \frac{1}{2}\int_{t_0(\varphi)+\varepsilon}^1 (\dot{\varphi}(s) - b(\varphi(s),\sigma_s))^2 ds - \int_{t_0(\varphi)}^{t_0(\varphi)+\varepsilon} \varphi(s)\dot{k}_\varphi(s)ds - \frac{1}{2}\int_0^\varepsilon k_\varphi^2(s)ds.$$

This last quantity tends to $I_0(\varphi)$ as ε tends to 0. □

5. The upper bound

We first prove the following

Proposition 5.1. *Let $\theta \ge 0$ and $\varphi \in C_\theta([0,1];\mathbb{R})$ such that there exists $t_0 \in [0,1]$ so that $\varphi(t_0) < 2\sqrt{t_0}$. Then*

$$\lim_{\delta\downarrow 0}\lim_{N\to\infty} \frac{1}{N}\ln \mathbb{P}(\lambda_1^{\theta,N} \in B(\varphi,\delta)) = -\infty.$$

We proceed as in Ben Arous et al. (2001). From Cabanal Duvillard and Guionnet (2001), we know that the process μ_N satisfies a LDP in the scale N^2 with a good rate function whose unique minimizer is the semicircular process σ for which we know that the support of σ_t is $[-2\sqrt{t}, 2\sqrt{t}]$.

Let $\delta_0 = 2\sqrt{t_0} - \varphi(t_0)$. By continuity of φ, there exists $\varepsilon > 0$, such that for any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, $\varphi(t) < 2\sqrt{t} - \frac{\delta_0}{2}$.
For any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, there exists f_t such that $f_t(y) = 0$ if $y \leq \varphi(t)$ and $\int f_t(x) d\sigma_t(x) > 0$. We let $F := \{\mu \in C([0,1]; \mathcal{P}(\mathbb{R}))/ \int f_t(x) d\mu_t(x) = 0 \; \forall t \in [t_0 - \varepsilon, t_0 + \varepsilon]\}$, which is a closed set.
For any $\delta < \frac{\delta_0}{2}$,

$$\mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq \mathbb{P}(\mu_N \in F).$$

As $\sigma \notin F$, $\limsup_{N\to\infty} \frac{1}{N^2} \ln \mathbb{P}(\mu_N \in F) < 0$, which gives the Proposition. □
We thus consider the case where $\varphi(t) \geq 2\sqrt{t}$ and as a first step, we prove the upper bound for a function φ which satisfies $\varphi(t) > 2\sqrt{t}$ for all $t \in [0,1]$ (this implies in particular that $\theta > 0$).

5.1. *The upper bound for functions φ well separated from $t \mapsto 2\sqrt{t}$.*

Proposition 5.2. *Let $\varphi \in C_\theta([0,1]; \mathbb{R})$ such that for any $t \in [0,1]$, $\varphi(t) > 2\sqrt{t}$. Then*

$$\lim_{\delta \downarrow 0} \limsup_{N\to\infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1^{\theta,N} \in B(\varphi, \delta)) \leq -I_\theta(\varphi). \tag{5.1}$$

The general idea to prove (5.1) is, as for the lower bound, to introduce the exponential martingales M_t^h (see (4.12) and (4.19)) and to optimize on h.

$$\mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq \mathbb{P}\left(\lambda_1 \in B(\varphi, \delta); \nu_N \in \mathbb{B}(\sigma, \alpha)\right) + \mathbb{P}(\nu_N \notin \mathbb{B}(\sigma, \alpha)) \tag{5.2}$$

and the second term is exponentially negligible in scale N since ν_N satisfies a large deviation principle in scale N^2 (this would be not true if we replace $\mathbb{B}(\sigma, \alpha)$ by $\mathbb{B}_r(\sigma, \alpha)$). Now,

$$\begin{aligned} \mathbb{P}\left(\lambda_1 \in B(\varphi, \delta); \nu_N \in \mathbb{B}(\sigma, \alpha)\right) &= \mathbb{E}\left(\mathbf{1}_{\lambda_1 \in B(\varphi,\delta); \nu_N \in \mathbb{B}(\sigma,\alpha)} \frac{M_1^h}{M_1^h}\right) \\ &\leq \exp\left(-N \inf_{(\psi,\mu) \in B(\varphi,\delta) \times \mathbb{B}(\sigma,\alpha)} F_N(\psi, \mu; h)\right). \end{aligned}$$

Unfortunately, the fonction F (or F_N) is not continuous on $B(\varphi, \delta) \times B(\sigma, \alpha)$ and we cannot conclude that

$$\lim_{\delta \to 0, \alpha \to 0} \inf_{(\psi,\mu) \in B(\varphi,\delta) \times \mathbb{B}(\sigma,\alpha)} F_N(\psi, \mu; h) = F(\varphi, \sigma; h).$$

Therefore, the strategy of the proof is first to prove that with high probability, only a finite number (say K) of eigenvalues can deviate strictly above $t \mapsto 2\sqrt{t}$ and then to introduce exponential martingales depending on $\lambda_1, \ldots, \lambda_K$ involving a functional $F(\lambda_1, \ldots, \lambda_K, \mu_N^{(1)}, \ldots, \mu_N^{(K)})$ (see (5.10)) which is now continuous on the sets that we are considering.
We first prove

Proposition 5.3. *For any $\eta > 0$, $L > 0$, there exists $K := K(\eta, L)$ (independent of N) such that*

$$\limsup_{N\to\infty} \frac{1}{N} \ln \mathbb{P}(\exists t \in [0,1], \lambda_{K+1}^{\theta,N}(t) > 2\sqrt{t} + \eta) \leq -L. \tag{5.3}$$

We will first prove a fixed time version of the same result stated in the following lemma:

Lemma 5.4. *For any $\eta > 0$, $L > 0$, there exists $K := K(\eta, L)$ such that for any $t \in [0,1]$,*

$$\limsup_{N\to\infty} \frac{1}{N} \ln \mathbb{P}(\lambda_{K+1}^{\theta,N}(t) > 2\sqrt{t} + \eta) \leq -L.$$

Proof: The first observation is that, as $H_N^\theta(t) = H_N^0(t) + \mathrm{diag}(\theta, 0, \dots, 0)$, with $\theta \geq 0$, by Weyl's inequalities, $\lambda_{K+1}^{\theta,N}(t) \leq \lambda_K^{0,N}(t)$ so that

$$\mathbb{P}(\lambda_{K+1}^{\theta,N}(t) > 2\sqrt{t} + \eta) \leq \mathbb{P}\left(\lambda_K^{0,N}(t) > 2\sqrt{t} + \eta\right) = \mathbb{P}\left(\lambda_K^{0,N}(1) > 2 + \frac{\eta}{\sqrt{t}}\right)$$
$$\leq \mathbb{P}(\lambda_K^{0,N}(1) > 2 + \eta).$$

Therefore, Lemma 5.4 will be a direct consequence of the fact that for any $p \geq 1$, the law of $(\lambda_1^{0,N}(1), \dots, \lambda_p^{0,N}(1))$ satisfies a LDP in the scale N with good rate function

$$F : (x_1, \dots, x_p) \mapsto 1_{x_1 \geq x_2 \geq \dots \geq x_p} \sum_{i=1}^p K_0(x_j),$$

with K_0 the individual rate function at time 1 as defined in Theorem 1.2. This is a particular case of Theorem 2.10 in Benaych-Georges et al. (2011) in the case when the potential V is just Gaussian ($V(x) = x^2$) therein.
From this, if we define $K \geq \frac{L}{K_0(2+\eta)}$, we deduce that,

$$\limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_{K+1}^{\theta,N}(t) > 2\sqrt{t} + \eta) \leq \limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_K^{0,N}(1) > 2 + \eta)$$
$$= \limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_1^{0,N}(1) > 2 + \eta, \dots, \lambda_K^{0,N}(1) > 2 + \eta)$$
$$\leq -K K_0(2 + \eta) \leq -L. \qquad \square$$

Proof of Proposition 5.3: We fix $\eta > 0$ and $L > 0$. Let R be such that $\frac{1}{10}\eta^2 R > 18L$ and choose a subdivision $(t_k)_{1 \leq k \leq R}$ of the interval $[0,1]$ such that for all $1 \leq k \leq R$, $|t_k - t_{k+1}| \leq \frac{2}{R}$. Now, for any $K \in \mathbb{N}^*$

$$\begin{aligned} \mathbb{P}(\exists t \in [0,1], \lambda_{K+1}(t) &> 2\sqrt{t} + \eta) = \mathbb{P}[\cup_k(\exists t \in [t_k, t_{k+1}], \lambda_{K+1}(t) > 2\sqrt{t} + \eta)] \\ &\leq R \max_{1 \leq k \leq R} \mathbb{P}(\exists t \in [t_k, t_{k+1}], \lambda_{K+1}(t) > 2\sqrt{t} + \eta). \end{aligned} \tag{5.4}$$

Then

$$\mathbb{P}(\exists t \in [t_k, t_{k+1}], \lambda_{K+1}(t) > 2\sqrt{t} + \eta) \leq \mathbb{P}\left(\lambda_{K+1}(t_k) > 2\sqrt{t_k} + \frac{\eta}{2}\right)$$
$$+\mathbb{P}\left(\exists t \in [t_k, t_{k+1}], \lambda_{K+1}(t) > 2\sqrt{t} + \eta\,;\, \lambda_{K+1}(t_k) \leq 2\sqrt{t_k} + \frac{\eta}{2}\right)$$
$$\leq \mathbb{P}\left(\lambda_{K+1}(t_k) > 2\sqrt{t_k} + \frac{\eta}{2}\right) + \mathbb{P}\left(\sup_{t_k \leq t < t_{k+1}} |\lambda_{K+1}(t) - \lambda_{K+1}(t_k)| \geq \frac{\eta}{3}\right).$$

From Lemma 5.4, we can find $K := K(\eta, L)$ such that

$$\limsup_N \frac{1}{N} \ln \mathbb{P}\left(\lambda_{K+1}(t_k) > 2\sqrt{t_k} + \frac{\eta}{2}\right) \leq -L.$$

From Lemma 2.2 applied for $p = K(\eta, L)$,

$$\limsup_N \frac{1}{N} \ln \mathbb{P}\left(\sup_{t_k \leq t < t_{k+1}} |\lambda_{K+1}(t) - \lambda_{K+1}(t_k)| \geq \frac{\eta}{3} \right) = -L.$$

As R is independent of N, (5.4) gives the lemma. □

Lemma 5.5. *Let K be fixed as in Proposition 5.3. Let $j \in \{1, \dots K\}$. We denote by $\mu_N^{(j)}$ the spectral measure of the $N - j$ smallest eigenvalues $\mu_N^{(j)} = \frac{1}{N-j} \sum_{p=j+1}^{N} \delta_{\lambda_p}$. Then,*

$$d(\mu_N^{(j)}, \mu_N) \leq \frac{2K}{N}.$$

Therefore, if $\mu_N \in \mathbb{B}(\sigma, \alpha)$, $\mu_N^{(j)} \in \mathbb{B}(\sigma, 2\alpha)$ for $N \geq N_0$.

Proof: Let $f \in \mathcal{F}_{Lip}$,

$$\mu_N^{(j)}(f) - \mu_N(f) = \frac{j}{N(N-j)} \sum_{p>j} f(\lambda_p) - \frac{1}{N} \sum_{p=1}^{j} f(\lambda_p)$$

and

$$|\mu_N^{(j)}(f) - \mu_N(f)| \leq \frac{j}{N} + \frac{j}{N} \leq \frac{2K}{N}.$$

□

Proof of Proposition 5.2: Let $\varphi \in C_\theta([0,1]; \mathbb{R})$ such that for any $t \in [0,1]$, $\varphi(t) > 2\sqrt{t}$. We recall that $r = \frac{1}{2} \inf(\varphi(t) - 2\sqrt{t})$

For $K \in \mathbb{N}^*$, let $\delta > 0$ such that $\delta < \frac{r}{4K}$ and $\alpha > 0$. We have

$$\mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq \mathbb{P}(A_{N,\delta,\alpha,K}) + \mathbb{P}(\exists t \in [0,1], \lambda_{K+1}(t) > 2\sqrt{t} + r) + \mathbb{P}(\mu_N \notin \mathbb{B}(\sigma, \alpha)) \tag{5.5}$$

with

$$A_{N,\delta,\alpha,K} := \left\{ \lambda_1 \in B(\varphi, \delta); \forall p > K, \forall t, \lambda_p(t) \leq 2\sqrt{t} + r;\ \mu_N \in \mathbb{B}(\sigma, \alpha) \right\}.$$

For each $t \in [0,1]$, if we are on $A_{N,\delta,\alpha,K}$, there exists at least a gap between two eigenvalues (among the K largest ones) larger than δ. To proceed, we will choose a subdivision $(t_k)_{1 \leq k \leq R}$ as in the proof of Proposition 5.3 and we will decompose the event $A_{N,\delta,\alpha,K}$ according to the location of the gap of size δ. More precisely, we have

$$A_{N,\delta,\alpha,K} \subset \bigcup_{\mathbf{i} \in \{1, \dots K\}^R} A_{N,\mathbf{i},\delta,\alpha} \tag{5.6}$$

where, for $\mathbf{i} = (i_1, \dots, i_R) \in \{1, \dots, K\}^R$,

$$\begin{aligned} A_{N,\mathbf{i},\delta,\alpha} = \{ & \lambda_1 \in B(\varphi, \delta), \forall k \leq R, \forall i < i_k, \lambda_i(t_k) - \lambda_{i+1}(t_k) \leq \delta, \\ & \lambda_{i_k}(t_k) - \lambda_{i_k+1}(t_k) > \delta, \mu_N \in \mathbb{B}(\sigma, \alpha) \}. \end{aligned}$$

As, for $i \leq i_k$, $\lambda_i(t_k) \geq \varphi(t_k) - i\delta$,

$$\begin{aligned} A_{N,\mathbf{i},\delta,\alpha} \subset \{ & \lambda_1 \in B(\varphi, \delta), \forall k \leq R, \forall i < i_k, \lambda_i(t_k) \geq \varphi(t_k) - i\delta, \\ & \lambda_{i_k}(t_k) - \lambda_{i_k+1}(t_k) > \delta, \mu_N \in \mathbb{B}(\sigma, \alpha) \}. \end{aligned}$$

Now, we choose R such that $\sup_{|t-s|\le\frac{2}{R}}|\varphi(s)-\varphi(t)|\le\frac{\delta}{6}$ and the subdivision such that $|t_k-t_{k+1}|\le\frac{2}{R}$. If we let

$$B_{N,\mathbf{i},\delta,\alpha}=\Big\{\lambda_1\in B(\varphi,\delta),\forall i<i_k,\forall t\in[t_k,t_{k+1}[,\lambda_i(t)\ge\varphi(t)-\left(i+\frac{1}{3}\right)\delta,$$
$$\lambda_{i_k}(t)-\lambda_{i_k+1}(t)>\frac{2}{3}\delta,\mu_N\in\mathbb{B}(\sigma,\alpha)\Big\},$$

then

$$A_{N,\mathbf{i},\delta,\alpha}\subset B_{N,\mathbf{i},\delta,\alpha}\bigcup\left\{\exists k,\exists t\in[t_k,t_{k+1}],\exists i\le i_k+1,|\lambda_i(t)-\lambda_i(t_k)|>\frac{\delta}{6}\right\}. \quad (5.7)$$

The second term will again be controlled by Lemma 2.2 and we now need to work on $B_{N,\mathbf{i},\delta,\alpha}$. Our goal is to show that for any $K\in\mathbb{N}^*$, any $h\in\mathcal{H}$, and any subdivision $(t_k)_{1\le k\le R}$ of $[0,1]$,

$$\lim_{\delta\to0}\lim_{\alpha\to0}\limsup\frac{1}{N}\ln\mathbb{P}(B_{N,\mathbf{i},\delta,\alpha})\le-F(\varphi,\sigma;h). \quad (5.8)$$

The idea is, as for the lower bound to make a change of measure given by a martingale but this time it will depend not only on λ_1 but on all the eigenvalues near φ above the gap of size δ. Then the average of these eigenvalues will be near φ and the variance of their Brownian part will be smaller than for an individual eigenvalue.

More precisely, let $j\le K$ and $X_j(t):=\frac{1}{j}\sum_{i=1}^j\lambda_i(t)$ is a solution of the SDE

$$dX_j(t)=\frac{1}{\sqrt{N}}\frac{1}{j}\sum_{i\le j}dB_i(s)+\frac{1}{N}\frac{1}{j}\sum_{i\le j}\sum_{p>j}\frac{1}{\lambda_i(t)-\lambda_p(t)}dt. \quad (5.9)$$

We denote by $dB^j(s)=\frac{1}{\sqrt{j}}\sum_{i=1}^j dB_i(s)$, which is a standard Brownian motion. Let $h\in\mathcal{H}$, we define the exponential martingale

$$\widetilde{M}_t^h=\exp\left[N\left(\int_0^t\sum_{k\le R}h(s)1_{[t_k,t_{k+1}[}(s)\frac{1}{\sqrt{N}}\frac{1}{\sqrt{i_k}}dB^{i_k}(s)\right.\right.$$
$$\left.\left.-\frac{1}{2}\int_0^t\sum_{k\le R}1_{[t_k,t_{k+1}[}(s)\frac{1}{i_k}h^2(s)ds\right)\right].$$

$$\begin{aligned}\frac{1}{\sqrt{N}}\frac{1}{\sqrt{i_k}}dB^{i_k}(t) &= dX_{i_k}(t)-\frac{1}{N}\frac{1}{i_k}\sum_{i\le i_k}\sum_{p>i_k}\frac{1}{\lambda_i(t)-\lambda_p(t)}dt\\ &= dX_{i_k}(t)-\frac{N-i_k}{N}\frac{1}{i_k}\sum_{i\le i_k}\int\frac{(\mu_N^{(i_k)})_t(dx)}{\lambda_i(t)-x}\,dt\end{aligned}$$

from the definition of the measures $\mu_N^{(i)}$ given in Lemma 5.5. Thus

$$\begin{aligned}\widetilde{M}_1^h &= \exp\left[N\sum_k\left([h_sX_{i_k}(s)]_{t_k}^{t_{k+1}}-\frac{N-i_k}{N}\frac{1}{i_k}\sum_{i\le i_k}\int_{t_k}^{t_{k+1}}\int\frac{\mu_t^{(i_k)}(dx)}{\lambda_i(t)-x}h(t)\,dt\right.\right.\\ &\left.\left.-\int_{t_k}^{t_{k+1}}\dot{h}(s)X_{i_k}(s)ds-\frac{1}{2i_k}\int_{t_k}^{t_{k+1}}h^2(s)ds\right)\right].\end{aligned}$$

We recall from Lemma 5.5 that if $\mu_N \in \mathbb{B}(\sigma,\alpha)$, $\mu_N^{(i_k)} \in \mathbb{B}(\sigma, 2\alpha)$.
$\widetilde{M}_1^h$ can be written as a functional

$$\widetilde{M}_1^h = \exp\left(NF_N(\lambda_1,\dots\lambda_K,\mu_N^{(1)},\dots,\mu_N^{(K)};h)\right). \tag{5.10}$$

We denote by

$$\begin{aligned}\Lambda_{\mathbf{i},\delta,\alpha} \quad = \quad & \{(\psi_1,\dots\psi_K,\nu_1,\dots\nu_K) : \psi_1 \in B(\varphi,\delta),\\ & \forall k \le R, \forall i < i_k, \forall t \in [t_k,t_{k+1}[, \psi_i(t) \ge \varphi(t) - \left(i+\frac{1}{3}\right)\delta,\\ & \psi_{i_k}(t) - \psi_{i_{k+1}}(t) > \frac{2}{3}\delta; \nu_i \in \mathbb{B}(\sigma,2\alpha), \mathrm{supp}(\nu_{i_k}(.)) \subset]-\infty, \psi_{i_k+1}(.)\}\end{aligned}$$

where in the above set, the functions are such that $\psi_1 \ge \psi_2 \dots \ge \psi_K$. We denote by $\underline{\psi} = (\psi_1,\dots\psi_K)$ and $\underline{\nu} = (\nu_1,\dots,\nu_K)$. Then,

$$\begin{aligned}\mathbb{P}(B_{N,\mathbf{i},\delta,\alpha}) \quad &= \quad \mathbb{E}\left[1_{B_{N,\mathbf{i},\delta,\alpha}} \frac{M_1^h}{M_1^h}\right]\\ &\le \quad \exp\left(-N \inf_{(\underline{\psi},\underline{\nu})\in\Lambda_{\mathbf{i},\delta,\alpha}} F_N(\underline{\psi},\underline{\nu};h)\right)\mathbb{E}[M_1^h]\\ &\le \quad \exp\left(-N \inf_{(\underline{\psi},\underline{\nu})\in\Lambda_{\mathbf{i},\delta,\alpha}} F_N(\underline{\psi},\underline{\nu};h)\right)\end{aligned}$$

and

$$\frac{1}{N}\ln\mathbb{P}(B_{N,\mathbf{i},\delta,\alpha}) \le -\inf_{(\underline{\psi},\underline{\nu})\in\Lambda_{\mathbf{i},\delta,\alpha}} F_N(\underline{\psi},\underline{\nu};h),$$

$$\limsup \frac{1}{N}\ln\mathbb{P}(B_{N,\mathbf{i},\delta,\alpha}) \le -\inf_{(\underline{\psi},\underline{\nu})\in\Lambda_{\mathbf{i},\delta,\alpha}} F_{\mathbf{i}}(\underline{\psi},\underline{\nu};h)$$

where

$$\begin{aligned}F_{\mathbf{i}}(\underline{\psi},\underline{\nu};h) \quad = \quad & \sum_k \Bigg[[h_s\Psi_{i_k}(s)]_{t_k}^{t_{k+1}} - \frac{1}{i_k}\sum_{i\le i_k}\int_{t_k}^{t_{k+1}}\int \frac{(\nu_{i_k})_t(dx)}{\psi_i(t)-x}h(t)\,dt\\ & - \int_{t_k}^{t_{k+1}} \dot{h}(s)\Psi_{i_k}(s)ds - \frac{1}{2i_k}\int_{t_k}^{t_{k+1}} h^2(s)ds\Bigg]\end{aligned}$$

with $\Psi_j = \frac{1}{j}\sum_{i\le j}\psi_i$.

Let us take $\alpha \to 0$. The function $\underline{\nu} \mapsto F_{\mathbf{i}}(\underline{\psi},\underline{\nu};h)$ is continuous on the set $\Lambda_{\mathbf{i},\delta,\alpha}$ since for $i \le i_k$,

$$\psi_i(t) - x \ge \frac{2}{3}\delta \quad \forall x \in \mathrm{supp}((\nu_{i_k})_t).$$

We obtain

$$\lim_{\alpha\to0}\limsup \frac{1}{N}\ln\mathbb{P}(B_{N,\mathbf{i},\delta,\alpha}) \le -\inf_{\underline{\psi}\in\Lambda_{\mathbf{i},\delta}} F_{\mathbf{i}}(\underline{\psi},\underline{\sigma};h)$$

where $\underline{\sigma} = (\sigma,\dots\sigma)$ and $\Lambda_{\mathbf{i},\delta}$ is defined as in $\Lambda_{\mathbf{i},\delta,\alpha}$ without the conditions on ν_i. Now, take $\delta \to 0$, the above functional is continuous in $\underline{\psi}$ and

$$\lim_{\delta\to0}\lim_{\alpha\to0}\limsup \frac{1}{N}\ln\mathbb{P}(B_{N,\mathbf{i},\delta,\alpha}) \le -F_{\mathbf{i}}(\underline{\varphi},\underline{\sigma};h) \tag{5.11}$$

where

$$F_{\mathbf{i}}(\underline{\varphi}, \underline{\sigma}; h) = h(1)\varphi(1) - h(0)\varphi(0) - \int_0^1 \int_{\mathbb{R}} \frac{\sigma_t(dx)}{\varphi(t) - x} h(t)dt - \int_0^t \dot{h}(s)\varphi(s)ds - \sum_k \frac{1}{2i_k} \int_{t_k}^{t_{k+1}} h^2(s)ds$$

and

$$-F_{\mathbf{i}}(\underline{\varphi}, \underline{\sigma}; h) \leq -F(\varphi, \sigma; h)$$

where F is defined by (3.2).

We have proved (5.8).

We now go back to the decompositions (5.6) and (5.7). Let us first treat the case when $I_\theta(\varphi) < \infty$. We choose $L = -2I_\theta(\varphi)$ and K as given in Proposition 5.3 so that

$$\limsup_N \frac{1}{N} \ln \mathbb{P}(\exists t \in [0, 1], \lambda_{K+1}(t) > 2\sqrt{t} + \eta) \leq -2I_\theta(\varphi).$$

Moreover

$$\limsup_N \frac{1}{N} \ln \mathbb{P}(\mu_N \notin \mathbb{B}(\sigma, \alpha)) = -\infty$$

and from Lemma 2.2, if we choose R, the number of points of the subdivision such that $R > \frac{260 I_\theta(\varphi)}{\delta^2}$,

$$\limsup_N \frac{1}{N} \ln \mathbb{P}\left(\exists k, \exists t \in [t_k, t_{k+1}], \exists i \leq i_k + 1 |\lambda_{i_k}(t) - \lambda_{i_k}(t_k)| > \frac{\delta}{6}\right) \leq -2I_\theta(\varphi).$$

We thus obtain, for any $h \in \mathcal{H}$,

$$\lim_{\delta \to 0} \limsup \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq -\inf(F(\varphi, \sigma; h), 2I_\theta(\varphi)).$$

Optimizing in h gives

$$\lim_{\delta \to 0} \limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq -I_\theta(\varphi).$$

In the case where $I_\theta(\varphi) = \infty$, for any L, we can associate K as in Proposition 5.3 such that

$$\limsup_N \frac{1}{N} \ln \mathbb{P}(\exists t \in [0, 1], \lambda_{K+1}(t) > 2\sqrt{t} + \eta) \leq -L. \tag{5.12}$$

In the same way as above, with $R > \frac{180L}{\delta^2}$, we then show that

$$\lim_{\delta \to 0} \limsup \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq -L$$

and since the left hand side does not depend on L,

$$\lim_{\delta \to 0} \limsup \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) = -\infty.$$

□

We now extend Proposition 5.2 to any function φ such that $\varphi(t) \geq 2\sqrt{t}$.

5.2. *The upper bound for functions φ not well separated from $t \mapsto 2\sqrt{t}$.*

Proposition 5.6. *Let $\varphi \in C_\theta([0,1];\mathbb{R})$ such that for any $t \in [0,1]$, $\varphi(t) \geq 2\sqrt{t}$. Then*

$$\lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1^\theta \in B(\varphi, \delta)) \leq -I_\theta(\varphi).$$

Proof of Proposition 5.6: For any $\epsilon > 0$, let $J_\epsilon = \{t \in [0,1], \varphi(t) > 2\sqrt{t} + \epsilon\}$. J_ϵ is an open set in $[0,1]$ and $\overline{J}_\epsilon$ is compact so that we can find a set V_ϵ of the form $V_\epsilon = \cup_{i=1}^{N_\epsilon}]a_i(\epsilon), b_i(\epsilon)[$ such that

$$\bar{J}_\epsilon \subset V_\epsilon \subset J_{\epsilon/2}.$$

Then, on V_ϵ, $\varphi(t) > 2\sqrt{t}$. For a function f on $[0,1]$, we denote by $f|_A$ its restriction to a subset A of $[0,1]$. Then,

$$\mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq \mathbb{P}(\lambda_1|_{V_\epsilon} \in B(\varphi|_{V_\epsilon}, \delta)).$$

From Proposition 5.2,

$$\lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1|_{V_\epsilon} \in B(\varphi|_{V_\epsilon}, \delta)) \leq -\sum_{i=1}^{N_\epsilon} I_\theta(\varphi|_{[a_i(\epsilon), b_i(\epsilon)]})$$

where

$$I_\theta(\varphi|_{[a,b]}) = \frac{1}{2} \int_a^b \left(\dot{\varphi}(s) - \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi(s)^2 - 4s} \right) \right)^2 ds,$$

this quantity may be infinite. Let $\epsilon \to 0$, by monotone convergence,

$$\lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq -\frac{1}{2} \int_J \left(\dot{\varphi}(s) - \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi(s)^2 - 4s} \right) \right)^2 ds$$

where $J = \{t \in [0,1], \varphi(t) > 2\sqrt{t}\}$ (the right-hand side can be infinite).

Assume that φ is differentiable almost everywhere (a.e.) on $[0,1]$. Since $\varphi(t) \geq 2\sqrt{t}$, if φ is differentiable in s_0 such that $\varphi(s_0) = 2\sqrt{s_0}$: then, $\dot{\varphi}(s_0) = \frac{1}{\sqrt{s_0}}$ and

$$\dot{\varphi}(s_0) - \frac{1}{2s_0} \left(\varphi(s_0) - \sqrt{\varphi(s_0)^2 - 4s_0} \right) = 0.$$

Thus,

$$\int_J \left(\dot{\varphi}(s) - \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi(s)^2 - 4s} \right) \right)^2 ds = I_\theta(\varphi).$$

If φ is not differentiable a.e., then $I_\theta(\varphi) = \infty$. Consider first the case $\theta > 0$. From the lower semicontinuity of I_θ, for all $C > 0$, there exists ϵ such that

$$B(\varphi, \epsilon) \subset \{\psi; I_\theta(\psi) > C\}.$$

Define

$$\begin{cases} \psi(t) = \varphi(t) & \text{on } \bar{\mathrm{J}}_\epsilon \\ \psi(t) = 2\sqrt{t} + \epsilon & \text{on } (\bar{\mathrm{J}}_\epsilon)^c \end{cases}$$

Then, $\psi \in B(\varphi, \epsilon)$ and

$$\int_{(\bar{J}_\epsilon)^c} \left(\dot{\psi}(s) - \frac{1}{2s} \left(\psi(s) - \sqrt{\psi(s)^2 - 4s} \right) \right)^2 ds$$
$$= \int_{(\bar{J}_\epsilon)^c} \left(\frac{1}{2s} \left(\epsilon - \sqrt{\epsilon^2 + 4\sqrt{s}\epsilon} \right) \right)^2 ds \leq K\epsilon,$$

for some constant K. The last inequality follows from the fact that since $\theta > 0$, $(\bar{J}_\epsilon)^c \subset [a,1]$ for a strictly positive a. Therefore, for ϵ small enough, $I_\theta(\psi|_{\bar{J}_\epsilon}) = I_\theta(\varphi|_{\bar{J}_\epsilon}) \geq \frac{C}{2}$. Moreover, $I_\theta(\varphi|_{V_\epsilon}) \geq I_\theta(\varphi|_{\bar{J}_\epsilon})$ so that we get

$$\lim_{\delta\downarrow 0} \limsup_{N\to\infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi,\delta)) \leq -\frac{C}{2}.$$

Since the inequality is true for all C,

$$\lim_{\delta\downarrow 0} \limsup_{N\to\infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi,\delta)) = -\infty.$$

Now for $\theta = 0$, if $I_0(\varphi|_{[a,1]}) < \infty$ for all $a > 0$, then, φ would be a.e. differentiable. Therefore, we can assume that there exists a a such that $I_0(\varphi|_{[a,1]}) = \infty$ and argue as before, using that $\mathbb{P}(\lambda_1 \in B(\varphi,\delta)) \leq \mathbb{P}(\lambda_1|_{[a,1]} \in B(\varphi|_{[a,1]},\delta))$. □

6. Contraction principle

The goal of this section is to get from Theorem 1.1 a new proof of Theorem 1.2 concerning the deviations of the largest eigenvalue at fixed time (say $t = 1$).

Note that this fixed time result has been used in the preceding section for the proof of the upper bound in the case $\theta = 0$, the goal here is to extend it to any $\theta > 0$.

Proof of Theorem 1.2 : As $\varphi \mapsto \varphi(1)$ is continuous, by contraction principle (Dembo and Zeitouni (2010, Theorem 4.2.1)), we get that $\lambda_1(1)$ satisfies a LDP with good rate function $J_{0,\theta}$, where, for any $\eta \in [0,1[$, we denote by

$$I_\eta(\varphi) = \int_\eta^1 f(t,\varphi(t),\dot{\varphi}(t))dt,$$

with

$$f(t,x,y) = \frac{1}{2}\left(y - \frac{1}{2t}\left(x - \sqrt{x^2-4t}\right)\right)^2$$

and, for $x \geq 2$, $\theta \geq 2\sqrt{\eta}$,

$$J_{\eta,\theta}(x) = \inf_{\substack{\varphi \text{ s.t. } \varphi(\eta)=\theta,\\ \varphi(1)=x}} I_\eta(\varphi).$$

As $I_\eta(\varphi)$ is a good rate function, the infimum in the above problem is reached, and we denote by φ^η an infimum. For a smooth function f, the classical theory of extremal problems (see eg. Ioffe and Tihomirov (1979)) predicts that φ^η should be solution of the Euler-Lagrange equation (6.2). Of course, the constraint $\varphi(t) \geq 2\sqrt{t}$ will play a crucial role and because the lack of smoothness of f around $x = 2\sqrt{t}$, the situation is more involved. However, most of the arguments used in the sequel are classical in such a context and we will only give sketches of the proofs.

We first show that the solution of Euler-Lagrange equation (6.2) realises the infimum among functions staying strictly above $t \mapsto 2\sqrt{t}$. Namely we have

Lemma 6.1. *For any $\eta \in]0,1[$, if, for any $t \in [\eta,1]$, $\varphi^\eta(t) > 2\sqrt{t}$, then,*

$$\varphi^\eta(t) = \frac{x-\theta}{1-\eta}(t-\eta) + \theta.$$

Proof: Let $\eta \in [0,1[$ be fixed.

It is easy to check that the infimum of I_η is finite, therefore, we know that φ^η is absolutely continuous with $\dot{\varphi}^\eta \in \mathbb{L}^1$. Following the proof of Theorem 4 in Chapter 9.2.3 in Ioffe and Tihomirov (1979) (the details are left to the reader), one can show that it is a solution to the DuBois-Reymond equation, i.e. there exists a constant r such that for any $t \in [\eta, 1]$,

$$\frac{\partial f}{\partial y}(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)) - \int_\eta^t \frac{\partial f}{\partial x}(s, \varphi^\eta(s), \dot{\varphi}^\eta(s)) = r. \tag{6.1}$$

From there, one can check that φ^η is a $\mathcal{C}^2$ solution of the Euler-Lagrange equation, that is, for any $t \in [\eta, 1]$,

$$\frac{d}{dt}\frac{\partial f}{\partial y}(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)) - \frac{\partial f}{\partial x}(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)) = 0. \tag{6.2}$$

Indeed, if we define

$$g(t, y) := f(t, \varphi^\eta(t), y) - y \int_\eta^t \frac{\partial f}{\partial x}(s, \varphi^\eta(s), \dot{\varphi}^\eta(s))ds - ry,$$

$g(t, .)$ is a convex quadratic polynomial, therefore it has a unique minimizer $y(t)$, which is solution of the equation

$$\frac{\partial g}{\partial y}(t, y(t)) = 0.$$

We can compute y explicitely, it is given by

$$y(t) = \frac{\varphi^\eta(t)}{2t} - \frac{1}{2t}\sqrt{\varphi^\eta(t)^2 - 4t} + \frac{1}{2t}\frac{\varphi^\eta(t)}{\sqrt{\varphi^\eta(t)^2 - 4t}} + r.$$

As we know that φ^η is absolutely continuous, so is y. But, by unicity of the minimizer, we have that $y = \dot{\varphi}^\eta$ so that we get that φ^η is continuously differentiable. Therefore $\frac{\partial g}{\partial y}$ is continuously differentiable in both variables and from the implicit function theorem, we get that $y(t)$ is continuously differentiable, so that φ^η is twice continuously differentiable. Differentiating (6.1) we get (6.2).

Straightforward computations leads to $\ddot{\varphi}^\eta \equiv 0$ and thus

$$\varphi^\eta(t) = \frac{x - \theta}{1 - \eta}(t - \eta) + \theta.$$

□

Now, the goal will be to determine in which cases this linear solution realises the infimum over all admissible functions and in which cases we can do better by touching the wall $t \mapsto 2\sqrt{t}$.

We need the comparison

Lemma 6.2. *For any $\eta \in [0,1[$, if there exists $t_0 \in [\eta, 1]$ such that $\varphi^\eta(t_0) = 2\sqrt{t_0}$, then*

$$I_\eta(\varphi^\eta) \geq \int_2^x \sqrt{u^2 - 4}\, du.$$

Proof: We have

$$I_\eta(\varphi^\eta) \geq \int_{t_0}^1 f(t, \varphi^\eta(t), \dot{\varphi}^\eta(t))dt.$$

For any φ such that $I_\eta(\varphi) < \infty$, for $t \geq \eta$, we denote

$$K_t(\varphi) = \frac{1}{2}\int_t^1 \left(\dot{\varphi}(s) - \frac{1}{2s}(\varphi(s) + \sqrt{\varphi^2(s) - 4s})\right)^2 ds.$$

If we let $y(s) = \frac{\varphi(s)}{\sqrt{s}}$, one has

$$K_t(\varphi) = I_t(\varphi) - \int_{y(t)}^{y(1)} \sqrt{u^2-4}\, du.$$

But $K_t \geq 0$ so that

$$I_\eta(\varphi^\eta) \geq \int_{\frac{\varphi^\eta(t_0)}{\sqrt{t_0}}}^x \sqrt{u^2-4}\, du = \int_2^x \sqrt{u^2-4} du.$$

□

From there, we can prove

Lemma 6.3. *For any $\eta \in [0, 1[$,*
if $\theta = 2\sqrt{\eta}$ or $\left(2\sqrt{\eta} < \theta < 1 + \eta \text{ and } x \leq \frac{\theta+\sqrt{\theta^2-4\eta}}{2} + \frac{2}{\theta+\sqrt{\theta^2-4\eta}}\right)$, then

$$J_{\eta,\theta}(x) = \int_2^x \sqrt{u^2-4}\, du.$$

Proof: The proof consists in exhibiting explicit functions φ^*_η realizing the infimum, that is such that $I_\eta(\varphi^*_\eta) = \int_2^x \sqrt{u^2-4} du$.
For $\theta = 2\sqrt{\eta}$ and $x \geq 2$, we let t^* be such that $\sqrt{t^*} := \frac{x+\sqrt{x^2-4}}{2}$ and

$$\varphi^*_\eta(t) = \begin{cases} 2\sqrt{t} & \text{if } \eta \leq t \leq t^* \\ 2\sqrt{t^*} + \frac{1}{\sqrt{t^*}}(t - t^*) & \text{if } t^* \vee \eta \leq t \leq 1 \end{cases}$$

For $2\sqrt{\eta} < \theta < 1 + \eta$ and $x \leq \frac{\theta+\sqrt{\theta^2-4\eta}}{2} + \frac{2}{\theta+\sqrt{\theta^2-4\eta}}$, we let s^* be such that $\sqrt{s^*} = \frac{\theta+\sqrt{\theta^2-4\eta}}{2}$, and

$$\varphi^*_\eta(t) = \begin{cases} \theta + \frac{1}{\sqrt{s^*}}(t-\eta) & \text{if } \eta \leq t \leq s^* \\ 2\sqrt{t} & \text{if } s^* \leq t \leq t^* \\ 2\sqrt{t^*} + \frac{1}{\sqrt{t^*}}(t - t^*) & \text{if } t^* \leq t \leq 1 \end{cases}$$

The extension to the case when $\eta = 0$ is easy to obtain and left to the reader. □

In the other cases, tedious computations allows to check that the solution of Euler-Lagrange realises the infimum. More precisely we have

Lemma 6.4. *For any $\eta \in]0, 1[$,*
if $\theta > 1 + \eta$ or $\left(2\sqrt{\eta} < \theta < 1 + \eta \text{ and } x > \frac{\theta+\sqrt{\theta^2-4\eta}}{2} + \frac{2}{\theta+\sqrt{\theta^2-4\eta}}\right)$, then

$$J_{\eta,\theta}(x) = I_\eta(d_{\eta,\theta,x}),$$

where for any $t \in [\eta, 1]$,

$$d_{\eta,\theta,x}(t) = \frac{(x-\theta)}{1-\eta}(t-\eta) + \theta.$$

The last point, to complete the proof of the Theorem is to extend the above lemma to the case when $\eta = 0$. More precisely, we have,

Lemma 6.5. *If $\theta \geq 1$ or $\left(0 < \theta < 1 \text{ and } x \geq \theta + \frac{1}{\theta}\right)$, let $\varphi_*(t) = (x-\theta)t + \theta$ for $t \in [0,1]$, then for any φ, $I_0(\varphi_*) \leq I_0(\varphi)$*

We do not detail the proof of the lemma which consists in cutting the integral defining I_θ, on $[0,\eta]$ with η small enough for the integral to be small and on $[\eta, 1]$ on which the previous lemmas give the minimizers.

This concludes the proof by the following computation. We set $\alpha = x - \theta$ and use the change of variable $u = \theta + \alpha t + \sqrt{(\theta+\alpha t)^2 - 4t}$.

$$\begin{aligned} I(\varphi_*) &= \frac{1}{2}\int_0^1 \left(\alpha - \frac{1}{2t}(\theta + \alpha t - \sqrt{(\theta+\alpha t)^2 - 4t})\right)^2 dt \\ &= -\ln\left(\frac{x+\sqrt{x^2-4}}{2\theta}\right) + \frac{1}{4}x\sqrt{x^2-4} + \frac{1}{4}x^2 - \theta x + \frac{\theta^2}{2} + \frac{1}{2}. \end{aligned}$$

This agrees with the formulae giving M_θ and L_θ since we have

$$\int_2^x \sqrt{z^2-4}\,dz = -2\ln\left(\frac{x+\sqrt{x^2-4}}{2\theta}\right) + \frac{1}{2}x\sqrt{x^2-4}. \qquad \square$$

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